SUBHARMONIC RESPONSE OF PASSIVE NETWORKS

CONTAINING NONLINEAR REACTIVE ELEMENTS

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SUMMARY

An analytical and experimental study of subharmonic resonance in nonlinear circuits containing ferroelectric capacitors is presented. Although ferromagnetic materials have been widely used to obtain nonlinear coils it is believed that this is the first reported application of ferroelectric materials to subharmonic generation. Due to the low loss of ferroelectric materials at radio frequencies networks incorporating ferroelectric elements are useful as subharmonic generators or frequency dividers at higher frequencies than are ferromagnetic nonlinear circuits.

The subharmonic circuits which use ceramic ferroelectric capacitors are almost linear and almost lossless. Analytical approximation methods are applied to obtain solutions of the differential equations which describe the response of such circuits. Methods are given for applying the perturbation series, Kryloff's first approximation and equivalent linearization methods to the study of the n-th order subharmonic resonance. The equivalent linearization and Kryloff's methods yield first approximations to the solution of an almost linear differential equation. Perturbation series is capable of yielding a higher approximation and the technique of computing a second approximation is given. A generalization of Kryloff's approximation method is derived which is capable of yielding successive higher approximations to the solutions of driven circuits. The second approximation is extensively developed by this successive approximation method for systems of one and two degrees of freedom. Methods of solving the systems of differential equations which describe the behavior of circuits of several degrees of freedom are also presented.

The Liapounoff first approximation method of investigation of the stability of solutions is discussed.

It is shown that solutions accurate to a second approximation are necessary to describe adequately the subharmonic response of circuits with small exciting forces unless there exists a resonant frequency near the excitation frequency. Thus the perturbation and successive approximation methods are most useful in the study of subharmonic resonance since they can yield second approximations to a solution. It is shown that the details of applying the successive approximation method are easier than the perturbation series. In particular, it is easy to investigate by Liapounoff's method the stability of solutions derived by the successive approximation method.

Most of the analytical solutions are presented for the case of small exciting forces. However, a change of variable is presented by which solutions can be calculated if the excitation is of order unity. The technique of computing such a solution is outlined.

Measurements of the incremental capacity and loss factor as a function of bias voltage were made for a nember of commercial ceramic titanate dielectrics. The results of these measurements are presented for the three dielectrics found to have the greatest nonlinearity. Measurements of the nonlinearity of the charge-voltage characteristic of the ferroelectric capacitors were made so as to accurately determine a polynomial approximation for this capacitor characteristic. Several methods of measurement were evolved to determine the coefficients of the polynomial approximation. Included in these methods were oscilloscope displays of charge versus voltage hysteresis loops, harmonic analysis of capacitor charge with a sine wave of voltage across the capacitor, and harmonic analysis of the capacitor voltage with a sinusoidal capacitor charge. The latter method was found to be the most satisfactory.

Polynomial coefficients computed from experimental data yield a satisfactory polynomial approximation to the capacitor characteristic for one value of capacitor alternating charge. This approximation is not accurate for a large range of capacitor charge or voltage amplitudes, since the capacitor characteristic varies with the amplitude of periodic variations about a bias point. In each case the multiple valued hysteresis loop of the capacitor is approximated by a single valued polynomial. Several hysteresis loops of charge versus voltage for different temperatures are presented to show the effects of temperature on the nonlinear characteristics of the capacitors.

The subharmonic resonance of second order in singly resonant circuits is analyzed by both the perturbation and successive approximation methods. The results derived by these methods differ only if the excitation frequency does not equal the circuit resonant frequency. Since this frequency difference was assumed in the analysis to be of the second order of smallness, the discrepancy is very small. A similar difference exists between the third order subharmonic solution computed by the successive approximation method and a previously published result derived by Duffing's iteration method.

A study of the stability of the second order subharmonic solution shows that for proper tuning conditions and sufficient excitation the subharmonic will build up from rest. The stability study also shows that if one-half the excitation frequency is less than the circuit resonant frequency, the input current or voltage required to initiate the subharmonic is greater than that required to sustain it.

A series of measurements are presented of the second order subharmonic amplitude as a function of input current amplitude and frequency. Some curves computed from the theoretical analysis are plotted with the measured results for comparison. These curves show reasonable agreement between theoretical and experimental results for the ranges of variables assumed in the analytical treatment. The theory presented does not predict the experimentally observed lower frequency limit of the subharmonic response. At this lower frequency limit the detuning is of greater order than was assumed in the analysis or the frequency was outside the range of variation assumed in the analysis. In the experimental work considerable care was necessary to minimize heating in the dielectric and in properly tuning the nonlinear circuit. Differences between experimental and computed results were found to be due largely to errors in tuning, errors in measurement of the circuit resistance, and errors in approximation of the nonlinear characteristic by a polynomial. The error in the second approximation to the solution of the differential equation seems to be less than the error resulting from the polynomial approximation to the characteristic of the nonlinear element, provided independent variables are held in the assumed ranges.

Waveforms are given of the second order subharmonic envelope response when the excitation is either periodically interrupted or sinusoidally amplitude-modulated. These waveforms show the envelope build-up and decay of the subharmonic. The subharmonic envelope is shown to be nearly a replica of the input modulation envelope for high modulation frequencies but distorted for low modulation frequencies. The highest second order subharmonic frequency obtained was 500 kc. This limit applies only for singly resonant circuits.

A general analysis is given of the n-th order subharmonic response of a circuit of a single degree of freedom. This analysis shows that subharmonics of third and higher orders cannot build up from rest conditions in singly resonant circuits. This result is confirmed experimentally.

The second order subharmonic response of a doubly resonant circuit is analyzed by Kryloff's first approximation method for the case in which one resonant frequency equals the excitation frequency and the other one-half the excitation frequency. If a significant third-degree curvature exists in the characteristic of the nonlinear element, there exist upper and lower limits of excitation amplitude for which the second order second order subharmonic exists. In such doubly resonant circuits the second order subharmonic is easily excited. Experimental results are presented on the subharmonic behavior with variation in excitation amplitude and frequency as well as characteristic curvature.

It is shown that in multiply resonant circuits responses can exist with frequencies which are not rational fractions of the input frequency. These irrational frequencies can synchronize with the frequency of a separate input signal. A second approximation analysis is derived for the irrational response of a doubly resonant circuit and experimental data are given on the amplitudes of the response as a function of amplitude of excitation. It is shown that irrational responses can exist in doubly resonant circuits with a small amplitude of excitation provided the sum of the resonant frequencies of the circuit is near the excitation frequency. Irrational responses can occur in doubly resonant circuits with large exciting forces or triply resonant circuits (the third resonant frequency equal to that of the excitation) with small excitation if the sum of two of the resonant frequencies of the circuit is a harmonic of the excitation frequency. The frequencies of irrational responses vary slightly with excitation amplitude. If irrational response frequency is near a subharmonic, the response may synchronize to become a stable subharmonic.

Since irrational responses can build up from rest and synchronize at a subharmonic of the excitation frequency, irrational responses can be used to cause subharmonics of greater than second order to be self starting. Analytical conditions are derived for which a response near a subharmonic frequency will build up from rest. If a subharmonic is excited under these conditions, there are generally values of input current amplitude and frequency for which the response is subharmonic and outside this range of response is irrational. However, for one dielectric used it was found possible to generate stable third and fourth order subharmonics which remain stable over all the region for which a response other than harmonic exists.

Experimental results are given of measurements on third, fourth and fifth order subharmonic responses of doubly resonant circuits. Waveforms of many higher order subharmonics are presented.

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CHAPTER I

INTRODUCTION

The currents which flow in electrical circuits driven by a current or voltage source can be classified according to the ratios of their frequencies to the frequency of the source. Three types of response--harmonic, subharmonic and irrational--can occur in nonlinear networks; that is, networks containing one or more current-or voltage-dependent elements. These types of response are defined as follows:

1. Harmonic--The response frequency is an integral multiple of the input frequency.

2. Subharmonic--The response frequency is $\frac{r}{s}$ times the input frequency, where r and s are integers ($r \neq ns$ where n is an integer).

3. Irrational--The ratio of the response frequency to the input frequency cannot be expressed as the ratio of two integers--that is, the ratio is an irrational number.

Subharmonic and irrational responses of passive electrical circuits were observed in the middle 1920's. In 1924, Heegner[1] reported a study of an iron-cored circuit, having two or three associated resonant circuits, in which both subharmonic and irrational responses were observed. Fallou[2], in 1926, carried out a study of the subharmonic response of a series RLC circuit containing a saturable core inductor. There followed a series of works in France and Germany on the properties of resonant circuits containing saturable inductors. These works have recently been summarized by Dehors[3].

Interest in subharmonic circuits has arisen as a result of their use in frequency changing circuits and their possible undesired existence in power distribution systems where capacitors are used for power-factor compensation. This latter consideration led to work by McCrumm[4] on the conditions for the existence of subharmonics in RLC series circuits.

U. S. patents on frequency changing, or dividing, circuits have been issued to Fallou[5], Heegner[6], McCreary[7] and Manley[8]. In 1936, a subharmonic generator[9] was manufactured for use as a ringing tone generator in telephone exchanges. Manley in his patent[8] and in a later paper[10] showed that subharmonics of order greater than two were not self starting in circuits of a single degree of freedom, but that they could be made so by the addition of another loop resonant at a multiple of the desired subharmonic frequency.

Recently Bennett[11] has published a review of nonlinear and varying parameter circuit theory. Cartwright[12] has published a survey of the general field of nonlinear vibrations including subharmonics. These survey articles give a number of references of historical interest. Minorsky[13] has shown that a subharmonic can be sustained in a network of linear reactive elements only if the network resistance is negative. That is, the circuit must be active. Thus subharmonic responses exist in passive circuits, only if the circuits contain nonlinear reactive elements.

Inductors with ferromagnetic cores were the only nonlinear reactive elements with low losses generally available until the discovery of the nonlinear properties of barium titanate. Wainer[14] and Wul[15] in 1946 reported that the relative dielectric constant of barium titanate was a function of the applied electric field intensity. Recently numerous papers have appeared on such nonlinear or ferroelectric dielectrics. Von Hippel[16], Roberts[17] and Jaynes[18] have developed theories of this class of dielectric. Merz[19] has studied the properties of single crystals of barium titanate. Roberts[20] has made extensive measurements of the nonlinear properties of barium titanate. Vincent[21], Shaw[22] and Urkuwitz[23] have reported on the use of ferroelectric materials in dielectric amplifiers. Hollman[24], Tucker[25] and Dranetz[26] have reported on applications of nonlinear dielectrics as modulators, and Anderson[27] has used single crystals of barium titanate as storage elements in computers.

Previous experimental investigations of subharmonics have used laminated iron or powdered iron as core materials for nonlinear coils. Since iron and powdered-iron core materials possess increased losses at frequencies above a few kilocycles per second, subharmonic investigations have been restricted to low frequencies. Ceramic ferromagnetic materials of the ferroxcube, ferrite, and ferramic types have recently been developed. Littman[28], Goldsmith[29] and Brockman[30] have published representative permeability and loss data on recently developed ferromagnetic materials. A search of the literature and manufacturers' data reveals that ferromagnetic materials are presently available either with low losses at radio frequencies or with high degrees of nonlinearity. However, the materials which saturate easily possess high losses at frequencies above one hundred kilocycles, and the materials which have low radio frequency losses become nonlinear only at high magnetomotive forces. Thus, nonlinear coils are useful as nonlinear reactive elements in applications which require high values of inductance or current. At high frequencies low or moderate values of inductance are required, and high currents are difficult to generate.

Since barium titanate capacitors possess fairly low losses up to frequencies of the order of fifty megacycles, ferroelectric capacitors are more suitable than ferromagnetic coils for use at high frequencies in subharmonic generators. The work of this thesis is concentrated upon the examination of properties of ferroelectric circuits. The analytical methods employed can be applied to circuits containing either nonlinear capacitors or inductors.

CHAPTER II

ANALYTICAL METHODS

Concurrently with the experimental interest in subharmonics, work on the applicable mathematical theory developed. The theory of the differential equations which describe the behavior of nonlinear systems has been the subject of many papers and a few recent books. Several methods of analysis of nonlinear systems have evolved. These methods can be grouped into four categories.

Analytical approximation methods. Several approximation methods have developed from the early limit cycle and perturbation series investigations of Poincaré and Lindstett. These methods are applicable to differential equations which possess periodic solutions and small nonlinearity. The perturbation series, successive approximations, and equivalent linearization methods are examples of these analytical methods. Most of the studies of nonlinear systems have used these approximate methods.

Topological methods. Topological methods provide a powerful tool for the study of the properties of nonlinear differential equations. Cartwright[31, 32] has reported a study of subharmonic response in active electrical circuits. However, topological methods thus far published have been restricted to second order differential equations.

Combinations of linear solutions. In a third method the nonlinear characteristic is approximated by a number of straight line segments. Linear solutions are found for each linear region, and the boundary conditions are chosen so as to yield a continuous solution. This method is capable of yielding accurate solutions but it is rather tedious to apply. Schouten[33] has used this method to investigate odd order subharmonics.

Numerical methods. A fourth category of methods includes graphical and numerical techniques of solution. Accurate solutions can be computed by numerical methods or obtained from computing machines. However, the influence of variations of a parameter is determined only if a solution is computed for each set of parameter values. Klotter[34] has reported an investigation of subharmonics using the numerical method of Ritz.

The analytical approximation methods of analysis are used in this study. These methods are used because they yield easily interpretable results and are applicable to rather general electrical circuits. The available ferroelectric materials possess small nonlinearities and low losses. Then the differential equations for the response of subharmonic circuits containing these materials will have small nonlinear and damping terms--that is, the differential equations are almost linear and almost conservative. Thus, the analytical approximation methods are applicable to the networks to be investigated.

Three analytical methods--perturbation series, Kryloff's approximations, and equivalent linearization--are used. The books by Minorsky[13], Stoker[35], McLachlan[36] and Kryloff[37] have treated these methods. A survey of the analytical literature, using the analytical approximation methods, is given below.

The perturbation series method has been used by Reuter[38] to study a second order differential equation with unsymmetrical nonlinear elasticity which may possess a second order subharmonic solution. This method has been used by Friedrichs[39] to study the third order subharmonic solution of a system with a single degree of freedom. Levenson[40] has used perturbations to investigate the subharmonic solutions of the Duffing equation.

The first approximation method of analysis has been used extensively in the study of subharmonic synchronization of oscillations. There has appeared no application of this method to passive current-or voltage-fed subharmonic circuits. Minorsky[41] and Bruce[42] have used this method to study the subharmonic response excited when a reactive circuit element is periodically varied. This type of excitation is termed parametric excitation. Bothwell[43] has recently published a generalization of this method of analysis to systems of many degrees of freedom.

A form of equivalent linearization has been used by Manley[10] in the study of subharmonics in saturable reactor circuits. Manley showed that in such circuits subharmonics of other than second order (one-half frequency) are not self starting in singly resonant circuits, but that higher order subharmonics can be made to build up from quiescent conditions by the addition of another loop to the circuit. This added loop must be resonant at or near a multiple of the subharmonic frequency. Manley states a condition for the build up from rest of the third order subharmonic[8], and he gives a detailed analysis of a singly resonant series circuit having a third order subharmonic response[10]. Ludeke[44] has used a method of linearization to discuss the harmonic and subharmonic response of a mechanical system with a nonlinear spring constant and a single degree of freedom. He also has published experimental results[45] on such a mechanical system.

The perturbation series and successive approximation methods are capable of yielding second or higher order approximations to the solution, while the equivalent linearization and Kryloff's first approximation methods give only a first approximation to the solution. A first approximation solution is generally adequate if there exists a sizeable current at the forcing frequency. So the first approximation method is useful if either the exciting force is large or there exists a resonant frequency near the exciting frequency.

The principal advantage of the Kryloff approximation methods is that the stability of the solution is easily investigated. The method of equivalent linearization possesses the advantages that a solution can be calculated without a detailed study of the differential equations of the circuit and that its results are given in terms more familiar to electrical engineers.

Second approximation solutions for the second order subharmonic response of a singly resonant circuit are derived in Chapter 4 by both the perturbation series and successive approximation methods. Kryloff's first approximation is applied in Chapter 5 to the study of the resonant excitation of a second order subharmonic. The successive approximation method is used in Chapter 5 to study irrational responses and the initiation of higher order subharmonics. Equivalent linearization is not applied to a specific problem but is felt to be of sufficient interest to warrant its inclusion in this chapter.

Details of the above three methods of analysis will be discussed separately in the remaining paragraphs of this section. It is the purpose of this material to present the elements of the general methods rather than particular engineering problems, which will be treated in subsequent sections.

Perturbation Series.

Following the method of Poincare' as presented by Stoker[35] or Minorsky[13], a periodic solution of a nonlinear differential equation which is almost linear and almost conservative (if such a solution exists) can be developed in a power series of a small parameter which describes the deviation from linearity and conservativeness. The technique consists of developing the solution as a power series in the small parameter and determining the series coefficients in such a manner that the solution will be periodic and will reduce to a known linear solution as the nonlinearity approaches zero.

According to Malkin[46] the construction of a periodic solution by perturbation series consists of three steps. First, a generating solution is found to which the desired solution reduces as the small parameter approaches zero. Second, the conditions under which there exist periodic solutions of the system corresponding to the generating solutions are determined. Third, the actual periodic solution which reduces to the generating solution with the vanishing of the parameter is calculated.

Obi[47] and Malkin[46] have considered problems in which the equation to which the system reduces with the vanishing of the parameter remains nonlinear and subharmonic resonance does not occur. This will not be discussed further here since the degree of nonlinearity obtained in the ceramic dielectrics is small, and the differential equations are almost linear. The generating functions are then solutions of linear differential equations.

The perturbation series used in this work follows Stoker[48]. If the forcing function is taken of order ϵ, a quasilinear differential equation which is frequently encountered in the analysis of circuits of a single degree of freedom is of the form

$$\frac{d^2y}{dt^2} + \Omega_o^2 y = \epsilon F \cos n\omega t - \epsilon f_1\left(\frac{dy}{dt}, y\right) - \epsilon^2 f_2\left(\frac{dy}{dt}, y\right) \quad , \qquad (1)$$

where ϵ is a small positive parameter, $n > 1$, f_1 and f_2 are expressible as polynomials in $\frac{dy}{dt}$ and y, and $\epsilon F \cos n\omega t$ is the forcing function.

Let $\theta = \omega t$.

Then,

$$\omega^2 y'' + \Omega_o^2 y \;=\; \epsilon F \cos n\theta - \epsilon F_1(\omega y', y) - \epsilon^2 f_2(\omega y', y) \;, \qquad (2)$$

where

$$y' = \frac{dy}{d\theta} \;,\; y'' = \frac{d^2y}{d\theta^2} \quad .$$

Now let it be assumed, for the moment, that a solution analytic in ϵ of period 2π exists. This solution by virtue of its analyticity can be expanded in a series in powers of ϵ.

Let the amplitude and frequency of the solution be expressed by the series

$$y = y_o + \epsilon y_1 + \epsilon^2 y_2 + \cdots \qquad (3)$$

and

$$\omega = \omega_o + \epsilon\,\omega_1 + \epsilon^2\omega_2 + \cdots$$

respectively.

Then if these series are substituted into (2) and terms of equal powers in ϵ are equated, a set of linear differential equations results. The solutions of these differential equations yield the coefficients ω_o, ω_1, --- and y_o, y_1 --- of (3). Equations (4), (5) and (6) are obtained by equating coefficients of like powers of ϵ^o, ϵ^1, and ϵ^2 respectively. Thus,

$$\omega_o^2 y_o'' + \Omega_o^2 y_o = 0 \ , \tag{4}$$

$$\omega_o^2 y_1'' + \Omega_o^2 y_1 = F\cos n\theta - f_1(\omega_o y_o' \ , y_o) - 2\,\omega_o\omega_1 y_o'' \tag{5}$$

and

$$\omega_o^2 y_2'' + \Omega_o^2 y_2 = -\ (\omega_1^2 + 2\omega_o\omega_2)y_o'' - (2\omega_o\omega_1 y_1'' - f_2(\omega_o y_o', y_o)$$

$$-\left[\frac{\partial}{\partial \dot{y}} f_1(\omega_o y_o', y_o)\right](\omega_o y_1' + \omega_1 y_o') - \frac{\partial}{\partial y} f_1(\omega_o y_o', y_o) y_1 \ , \tag{6}$$

where $\dot{y} = \frac{dy}{dt}$.

If a solution of period 2π exists, it follows from (4) that $\Omega_o = \omega_o$, $y_o = A\cos\theta + B\sin\theta$. The substitution of these results into (5) yields

$$y_1'' + y_1 = \frac{F}{\omega_o^2} \cos n\theta - \frac{f_1}{\omega_o^2}(-\omega_o A\sin\theta + \omega_o B\cos\theta, \tag{7}$$

$$A\cos\theta + B\sin\theta) + 2\frac{\omega_1}{\omega_o}(A\cos\theta + B\sin\theta) \ .$$

If $f_1(\omega_o y_o', y_o) = \sum_1^s (H_m \cos m\theta + K_m \sin m\theta)$,

then (6) becomes

$$y_1'' + y_1 = \frac{F}{\omega_o^2} \cos n\theta - \frac{1}{\omega_o^2}\sum_1^s (H_m \cos m\theta + K_m \sin m\theta) \tag{8}$$

$$+ 2 \frac{\omega_1}{\omega_o} (A\cos\theta + B\sin\theta) .$$

In order that y_1 be periodic, terms in $\cos\theta$ and $\sin\theta$ on the right hand side of (8) must sum to zero so that exceptional terms of the form $\theta\sin\theta$ or $\theta\cos\theta$ do not appear in y_1. Thus,

$$\left.\begin{aligned} &2\omega_o\omega_1 A\cos\theta = H_1(A, B)\cos\theta , \\ \text{and} \\ &2\omega_o\omega_1 B\sin\theta = K_1(A, B)\sin\theta . \end{aligned}\right\} \tag{9}$$

If the conditions given by (9) are satisfied, the solution of (8) is

$$y_1 = \frac{F}{\omega_o^2(1 - n^2)} \cos n\theta + A_1\cos\theta + B_1\sin\theta - \frac{1}{\omega_o^2}\sum_2^s\left[\frac{H_m}{1 - m^2}\cos m\theta + \frac{K_m}{1 - m^2}\sin m\theta\right] . \tag{10}$$

In equation (10), A_1 and B_1 are determined by initial conditions. For example, if at $t = 0$, $y(0) = A$, $y'(0) = B$, $y_1(0) = 0$, $y_1'(0) = 0$, A_1 and B_1 are fixed by the relations

$$\left.\begin{aligned} &0 = A_1 + \frac{F}{\omega_o^2(1 - n^2)} - \frac{1}{\omega_o^2}\sum_2^s \frac{H_m}{1 - m^2} \\ \text{and} \\ &0 = B_1 - \frac{1}{\omega_o^2}\sum_2^s \frac{m K_m}{1 - m^2} . \end{aligned}\right\} \tag{11}$$

However, it is often more convenient to set $A_1 = B_1 = 0$ and let the actual initial conditions differ slightly from A and B. This latter procedure will be followed.

Since y_o and y_1 are known, these can be substituted into (6) to determine y_2, and hence from (3) a second approximation to the solution. The conditions that y_2 be periodic yield two equations similar to (9), which fix the amplitude of the solution to the order of ϵ^2 since ω_2 is determined by (3) in terms of ω and the quantities ω_o, ω_1 and ϵ. If one sets $A_1 = B_1 = 0$, and

$$\sum_1^s \left[M_m(A, B)\cos m\theta + N_m(A, B)\sin m\theta \right] = f_2 + (\omega_o y_1' + \omega_1 y_o')\frac{\partial f_1}{\partial \dot{y}} + y_1 \frac{\partial f_1}{\partial y},$$

equation (6) becomes

$$y_2'' + y_2 = \left(\frac{\omega_1^2}{\omega_o^2} + 2\frac{\omega_2}{\omega_o} \right)(A\cos\theta + B\sin\theta) + 2\frac{\omega_1}{\omega_o^3}\frac{F\cos\theta}{(1 - n^2)} - \frac{2\omega_1}{\omega_o^3}\sum_2^s \left[\frac{H_m}{1 - m^2}\cos m\theta + \frac{K_m}{1 - m^2}\sin m\theta \right] - \frac{1}{\omega_o^2}\sum_1^s \left[H_m(A, B)\cos m\theta + N_m(A, B)\sin m\theta \right]. \tag{12}$$

The conditions that (12) possess a periodic solution are

$$P(A, B) = \left(\frac{\omega_1^2}{\omega_o^2} + 2\frac{\omega_2}{\omega_o} \right)A - \frac{1}{\omega_o^2} M_1(A, B) = 0$$

$$Q(A, B) = \left(\frac{\omega_1^2}{\omega_o^2} + 2\frac{\omega_2}{\omega_o} \right)B - \frac{1}{\omega_o^2} N_1(A, B) = 0 \tag{13}$$

The equations (13) fix the A, B amplitudes of the solution valid to an order of ϵ. The ω_1 and ω_2 values are determined from (9) and the frequency series of (3).

Thus far the existence of the subharmonic solution has been assumed but not proven to exist. If there exist values of A and B such that $P = Q = 0$ and that the Jacobian $J\left(\frac{P,Q}{A,B}\right) \neq 0$ for this A, B then by the implicit function theorem[35] a solution of (13) exists. These values of A and B are amplitudes of an n-th order subharmonic solution of (1). It should be noted that the method of construction of the solution assures that the solution approaches the generating function as $\epsilon \rightarrow 0$.

The calculation and existence of a subharmonic solution of (1) have been treated above by perturbation series. There is no assurance that these solutions will be stable. This question of stability can be treated by perturbing the periodic solutions; the stable solutions are then those solutions which return to the unperturbed values with increasing time. If this perturbed solution is substituted into the original second order differential equation and only first order perturbation terms retained, a linear differential equation of the Hill or Mathieu type results. The stability of the unperturbed solution is then determined by the stability or instability of the solutions of the corresponding linear Hill or Mathiew equation. This method of investigating stability is treated by Minorsky[13], Stoker[35] and McLachlan[36]. A second perturbation method of determining the stability of a solution consists of forming for the solution and its first derivative the differences between the initial values and the values one period later. A stable solution exists if this difference is such as to approach zero with time. This technique was used by Reuter[38].

In certain problems, for example the third order subharmonic, it is convenient to consider the forcing function to be of order unity rather than of small order. The perturbation series method can still be applied but the generating function consists of two terms, one subharmonic, and one at the forcing frequency. That is, if

$$\omega^2 y'' + \Omega^2 y = F \cos n\,\theta - \epsilon f_1(\omega y', y) \tag{14}$$

then the zeroth approximation or generating function is the solution of

$$\omega_o{}^2 y_o'' + \Omega^2 y_o = F \cos n\,\theta \,, \tag{15}$$

or

$$y_o = A \cos\theta + B \sin\theta + \frac{F}{\Omega^2(1 - n^2)} \cos n\,\theta \,. \tag{16}$$

The remainder of the solution is determined as before but with the new y_o. There is no known general rule to determine whether the forcing function should be taken of order unity or order small; both cases will arise later in this thesis. Generally the choice is made on the basis of physical magnitudes and ease of calculation of the solution.

It is clear that the perturbation series technique can be applied to systems of two or more degrees of freedom, provided such systems are almost linear and almost lossless. In systems of several degrees of freedom each amplitude and response frequency is expressed as a power series in the small parameter. Thus in a doubly resonant system there are four series whose coefficients are to be determined.

Kryloff and Bogoliuboff Approximation Methods.

In a series of studies, which have been translated by Minorsky[13] and Lefschetz[37], Kryloff and Bogoliuboff developed a method of calculation of successive approximations to the solution of a quasilinear differential equation. In this method the amplitude and instantaneous phase of the solutions are regarded as unknown and two first order differential equations are derived to describe their properties.

First approximation--This section deals with what is termed the first approximation and the next section shows how the second approximation can be derived. The discussion of the first approximation starts conveniently with a consideration of a second order quasilinear differential equation of the form

$$\ddot{q} + \omega^2 q = \epsilon H \cos n\omega t - \epsilon f(q, \dot{q}) , \qquad (17)$$

where ϵ is a small positive parameter, and $f(q, \dot{q})$ is a polynomial in q and $\dot{q}$. Also $f(q, \dot{q})$ includes a term of the form $(\Omega^2 - \omega^2)q$. It is not necessary that the forcing function be as simple as taken here but the form of (17) is sufficient for this thesis.

If $\epsilon = 0$, (1) reduces to the simple linear system having a solution of the form

$$\begin{aligned} q &= a \sin(\omega t + \phi) \\ \dot{q} &= \omega a \cos(\omega t + \phi) \end{aligned} \qquad (18)$$

In the case of ϵ not zero but small, the solution has the same form, but a and ϕ are functions of time yet to be determined. Note that $q = a \sin(\omega t + \phi)$ is the generating function of the perturbation method.

If the expression for q of (18) is differentiated, the result is

$$\dot{q} = \dot{a} \sin(\omega t + \phi) + a\omega \cos(\omega t + \phi) + a\dot{\phi} \cos(\omega t + \phi) .$$

So in order to satisfy the second equation of (18), it is necessary that

$$\dot{a} \sin(\omega t + \phi) + a\dot{\phi} \cos(\omega t + \phi) = 0 . \qquad (19)$$

Differentiation of $(\dot{q})$ from (18) gives $\ddot{q}$. Substitution of $\ddot{q}$ and q into

(17) gives

$$\dot{a}\omega \cos(\omega t + \phi) - a\omega\dot{\phi}\sin(\omega t + \phi) = \epsilon H \cos n\omega t$$
$$- \epsilon f\Big[a\sin(\omega t + \phi),\ a\omega \cos(\omega t + \phi)\Big] \quad . \tag{20}$$

When (19) and (20) are solved for $\dot{a}$ and $\dot{\phi}$, the resulting equations are

$$\dot{\phi} = -\frac{\epsilon}{a\omega}\Big\{H\cos n\omega t - f\Big[a\sin(\omega t + \phi), a\omega \cos(\omega t + \phi)\Big]\Big\} \sin(\omega t + \phi) \tag{21}$$

$$\dot{a} = \frac{\epsilon}{\omega}\Big\{H\cos n\omega t - f\Big[a\sin(\omega t + \phi),\ a\omega \cos(\omega t + \phi)\Big]\Big\} \cos(\omega t + \phi). \tag{22}$$

Equations (21) and (22) are now to be solved for the amplitude a and phase ϕ, which are time functions. It will be assumed that the variation with time of a and ϕ is slow compared to ωt. That is, the variation of a or ϕ in a period $\frac{2\pi}{\omega}$ is small so that a and ϕ can be determined by a quasi-steady-state analysis in which the right hand sides of (21) and (22) are replaced by their time averages. Then

$$\left.\begin{aligned} \dot{a} &= -\frac{\epsilon}{2\pi\omega}\int_0^{2\pi}\Big\{f\Big[a\sin(\omega t + \phi), \omega a \cos(\omega t + \phi)\Big] - H\cos n\omega t\Big\} \\ &\qquad\qquad \times\cos(\omega t + \phi)\ d(\omega t)\ , \\ &\text{and} \\ \dot{\phi} &= \frac{\epsilon}{2\pi a\omega}\int_0^{2\pi}\Big\{f\Big[a\sin(\omega t + \phi),\ \omega a \cos(\omega t + \phi)\Big] - H\cos n\omega t\Big\} \\ &\qquad\qquad \times \sin(\omega t + \phi) d(\omega t)\ . \end{aligned}\right\} \tag{23}$$

The equations (23) define the amplitude and phase of the first approximation to the subharmonic solution of (17) when ϵ is small. The subharmonic amplitude, a, and phase, ϕ, are solutions of the equilibrium conditions $\dot{a} = \dot{\phi} = 0$ of (23), provided $J\left(\frac{\dot{a}, \dot{\phi}}{a, \phi}\right) \neq 0$ for these values of a and ϕ.

Minorsky[49] introduced a change of variable that is useful where the forcing function is of order unity. The quasilinear differential equation is of the form

$$\ddot{q} + \omega^2 q = M\cos n\omega t - \epsilon f(q, \dot{q}) \tag{24}$$

A new variable, x, defined by $q = x + \frac{M}{\omega^2(1 - n^2)} \cos n\omega t$, is introduced into (24), which becomes

$$\ddot{x} + \omega^2 x = -\epsilon f\left[x + \frac{M}{\omega^2(1 - n^2)} \cos n\omega t\ ,\ \dot{x} - \frac{n M}{\omega(1 - n^2)} \sin n\omega t\right] . \tag{25}$$

The amplitude and phase of the first approximation to a solution of (24) are the solutions of $\dot{a} = \dot{\phi} = 0$, where

$$\left.\begin{aligned} \dot{a} &= -\frac{\epsilon}{2\pi\omega}\int_0^{2\pi} f\left[a\sin(\omega t + \phi) + \frac{M}{\omega^2(1 - n^2)} \cos n\omega t\ ,\right. \\ &\qquad \left. \omega a\cos(\omega t + \phi) - \frac{nM}{\omega(1 - n^2)}\sin \omega t\right]\cos(\omega t + \phi)d(\omega t) \\ \dot{\phi} &= \frac{\epsilon}{2a\omega\pi}\int_0^{2\pi} f\left[a\sin(\omega t + \phi) + \frac{M}{\omega^2(1 - n^2)} \cos n\omega t\ ,\right. \\ &\qquad \left. \omega a \cos(\omega t + \phi) - \frac{nM}{\omega(1 - n^2)} \sin \omega t\right] \sin(\omega t + \phi)d(\omega t) . \end{aligned}\right\} \tag{26}$$

Higher approximations--Although the first approximations of (23) and (26) are adequate to treat many quasilinear problems there remain many others, in which a second approximation is necessary or desirable. Higher approximations to the solution of (17) can be derived by an extension of Minorsky's work[50]. This extension is developed below.

It is assumed that (17) possesses a solution

$$q = Z(\psi, a) = Z(\theta, \phi, a) ,$$

where

$$\psi = \theta + \phi = \omega t + \phi$$

and

$$\omega \frac{da}{d\theta} = A(a, \phi), \ \omega \frac{d\psi}{d\theta} = \omega + \omega \frac{d\phi}{d\theta} = \omega + \sigma(a, \phi)$$

as in the firxt approximation (23). It is further assumed that expansions of Z, A, and σ exist in terms of the small parameter ϵ, such that

$$Z(\theta, \phi, a) = Z_o(\theta, \phi, a) + \epsilon Z_1(\theta, \phi, a) + \epsilon^2 Z_2(\theta, \phi, a) + --- \quad (27A)$$

$$\omega \frac{da}{d\theta} = A(a, \phi) = \epsilon A_1(a, \phi) + \epsilon^2 A_2(a, \phi) + --- \quad (27B)$$

$$\omega \frac{d\phi}{d\theta} = \sigma(a,\phi) = \epsilon \sigma_1(a, \phi) + \epsilon^2 \sigma_2(a, \phi) + --- . \quad (27C)$$

These expansions should be compared to those of the perturbation series method. It is seen that, in effect, one is treating the same problem but in this case the phase and amplitude of the solution are determined directly rather than the solution and its first derivative. In terms of the variable θ and an added term of order ϵ^2 equation (17) is

$$\omega^2 q'' + \omega^2 q = \epsilon H \cos n\theta - \epsilon f_1(q, \omega q) - \epsilon^2 f_2(q, \omega q) , \quad (28)$$

where primes indicate differentiation with respect to θ. By implicit differentiation one obtains

$$\frac{dq}{d\theta} = \frac{\partial Z}{\partial \theta} + \frac{\partial Z}{\partial \phi}\frac{\partial \phi}{\partial \theta} + \frac{\partial Z}{\partial a}\frac{\partial a}{\partial \theta} = \frac{\partial Z}{\partial \theta} + \frac{\sigma(a)}{\omega}\frac{\partial Z}{\partial \phi} + \frac{A(a)}{\omega}\frac{\partial Z}{\partial a}$$

and

$$\frac{d^2q}{d\theta^2} = \frac{\partial}{\partial \theta}\left(\frac{dq}{d\theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{dq}{d\theta}\right)\frac{\partial \phi}{\partial \theta} + \frac{\partial}{\partial a}\left(\frac{dq}{d\theta}\right)\frac{\partial a}{\partial \theta} = \frac{\partial^2 Z}{\partial \theta^2}$$

$$+ 2\frac{\sigma}{\omega}\frac{\partial^2 Z}{\partial \phi \partial \theta} + 2\frac{A}{\omega}\frac{\partial^2 Z}{\partial a \partial \theta} + \frac{\sigma^2}{\omega^2}\frac{\partial^2 Z}{\partial \phi^2} + 2\frac{\sigma}{\omega}\frac{A}{\omega}\frac{\partial^2 Z}{\partial a \partial \phi}$$

$$\frac{A^2}{\omega^2}\frac{\partial^2 Z}{\partial a^2} + \frac{\sigma}{\omega^2}\frac{\partial Z}{\partial \phi}\frac{\partial \sigma}{\partial \phi} + \frac{\sigma}{\omega^2}\frac{\partial Z}{\partial a}\frac{\partial A}{\partial \phi} + \frac{A}{\omega^2}\frac{\partial Z}{\partial \phi}\frac{\partial \sigma}{\partial a} + \frac{A}{\omega^2}\frac{\partial Z}{\partial a}\frac{\partial A}{\partial a}$$

Now if equations (27) are substituted into $\frac{dq}{d\theta}$ and $\frac{d^2q}{d\theta^2}$ and these are in turn substituted into (28), there results a differential equation involving terms of powers of ϵ. Equations (29), (30), (31) result from equating terms of ϵ^o, ϵ^1, and ϵ^2 respectively. These equations are

$$\frac{\partial^2 Z_o}{\partial \theta^2} + Z_o = 0 \ , \tag{29}$$

$$\frac{\partial^2 Z_1}{\partial \theta^2} + Z_1 = \frac{H}{\omega^2}\cos n\theta - \frac{2}{\omega}\sigma_1\frac{\partial^2 Z_o}{\partial \phi \partial \theta} \quad \frac{2}{\omega}A_1\frac{\partial^2 Z_o}{\partial a \partial \theta} - \frac{f_1}{\omega^2}\left(Z_o, \omega\frac{\partial Z_o}{\partial \theta}\right) \ , \tag{30}$$

and

$$\frac{\partial^2 Z_2}{\partial \theta^2} + Z_2 = -\frac{2}{\omega}\left(\sigma_2\frac{{}^2 Z_o}{\phi \quad \theta} + \sigma_1\frac{{}^2 Z_1}{\phi \quad \theta}\right) - \frac{2}{\omega}\left(A_2\frac{\partial^2 Z_o}{\partial a \partial \theta}\right. \tag{31}$$

$$+ A_1\frac{\partial^2 Z_1}{\partial a \partial \theta} - \frac{\sigma_1^2}{\omega^2}\frac{\partial^2 Z_o}{\partial \phi^2} - 2\frac{\sigma_1 A_1}{\omega^2}\frac{\partial^2 Z_o}{\partial a \partial \phi} - \frac{A_1^2}{\omega^2}\frac{\partial^2 Z_o}{\partial a^2}$$

$$- \frac{\sigma_1}{\omega^2}\frac{\partial Z_o}{\partial \phi}\frac{\partial \sigma_1}{\partial \phi} - \frac{\sigma_1}{\omega^2}\frac{\partial Z_o}{\partial a}\frac{\partial A_1}{\partial \phi} - \frac{\sigma_1}{\omega^2}\frac{\partial Z_o}{\partial \phi}\frac{\partial \sigma_1}{\partial a} - \frac{A_1}{\omega^2}\frac{\partial Z_o}{\partial a}\frac{\partial A_1}{\partial a}$$

$$- \frac{1}{\omega^2} \frac{\partial f_1}{\partial q} (q, \omega q') \Big| Z_1 - \frac{1}{\omega^2} \frac{\partial f_1}{\partial q'} (q, \omega q') \Big| \Big[\frac{\partial Z_1}{\partial \theta} + \frac{\sigma_1}{\omega} \frac{\partial Z_o}{\partial \phi} + \frac{A_1}{\omega} \frac{\partial Z_o}{\partial a} \Big] - \frac{1}{\omega^2} f_2(Z_o, \omega \frac{\partial Z_o}{\partial \theta}) .$$

In (31), $\frac{\partial f_1}{\partial q}, \frac{\partial f_1}{\partial q'}$ are evaluated at the point $Z_o, \frac{\omega \partial Z_o}{\partial \theta}$. The same process which yielded equations (29), (30), (31) can be extended to yield approximations of any desired order, but the second approximation is sufficient for the analyses of this thesis. A justification of equations (29), (30), (31) could be carried out by formal analogy with the perturbation series method but is not done herein. Rather, the two methods will be shown to yield solutions which are consistent to the same degree of approximation.

Equation (29) has a solution

$$Z_o = a\sin(\theta + \phi) , \tag{32}$$

which corresponds to the zeroth approximation of equation (18). Substitution of (32) into (31) yields

$$\frac{\partial^2 Z_1}{\partial \theta^2} + Z_1 = \frac{H}{\omega^2} \cos n\theta + \frac{2\sigma_1}{\omega} a\sin(\theta + \phi) - 2 \frac{A_1}{\omega} \cos(\theta + \phi) - \frac{f_1}{\omega^2} \Big[a\sin(\theta + \phi), \omega a\cos(\theta + \phi) \Big] . \tag{33}$$

In order that Z_1 be periodic, it is necessary that the terms involving $\sin \theta$ and $\cos \theta$ on the right hand side of (33) vanish. Let

$$f_1(Z_o, \omega \frac{\partial Z_o}{\partial \theta}) = \sum_{1}^{s} F_m(a)\sin m(\theta + \phi) + G_m(a)\cos m(\theta + \phi) .$$

Then the condition that Z_1 be periodic is

$$0 = 2\frac{\sigma_1 a}{\omega}\sin(\theta + \phi) - 2\frac{A_1}{\omega}\cos(\theta + \phi) - \frac{F_1(a)}{\omega^2}\sin(\theta + \phi)$$
$$- \frac{G_1(a)}{\omega^2}\cos(\theta + \phi) .$$

Or,

$$\left.\begin{aligned} A_1 &= -\frac{G_1(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f_1(Z_o, \omega\frac{\partial Z_o}{\partial \theta})\cos(\theta + \phi)d(\theta + \phi) \\ \sigma_1 &= \frac{F_1(a)}{2\omega a} = \frac{1}{2\pi\omega a}\int_0^{2\pi} f_1(Z_o, \omega\frac{\partial Z_o}{\partial \theta})\sin(\theta + \phi)d(\theta + \phi) . \end{aligned}\right\} \quad (34)$$

If equations (34) are substituted into the (27B) and (27C) equations and terms of powers greater than the first order in ϵ are dropped the first approximation (23) is obtained. When σ_1 and A_1 are given by (34), provided $n \neq 1$,

$$Z_1 = \frac{H\cos n\theta}{\omega^2(1 - n^2)} - \frac{1}{\omega^2}\sum_2^s\left[\frac{F_m(a)}{1 - m^2}\sin m(\theta + \phi)\right.$$
$$\left. + \frac{G_m(a)}{1 - m^2}\cos m(\theta + \phi)\right] . \quad (35)$$

With Z_o, and Z_1 known, a solution of (31) can be found, and A_2, σ_2 can be determined so that Z_2 is periodic. In (34)

$$\frac{\partial \sigma_1}{\partial \phi} = \frac{\partial A_1}{\partial \phi} = 0 .$$

If equation (32) is substituted into (31), it becomes

$$\frac{\partial^2 Z_2}{\partial\theta^2} + Z_2^2 = \frac{2\sigma_2}{\omega} a\sin(\theta + \phi) - \frac{2\sigma_1}{\omega}\frac{\partial^2 Z_1}{\partial\phi\partial\theta} - \frac{2A_2}{\omega}\cos(\theta + \phi) \tag{36}$$

$$- \frac{2A_1}{\omega}\frac{\partial^2 Z_1}{\partial\phi\partial\theta} + \frac{\sigma_1^2}{\omega^2} a\sin(\theta + \phi) - \frac{2\sigma_1 A_1}{\omega^2}\cos(\theta + \phi)$$

$$- \frac{\sigma_1}{\omega^2} a \frac{\partial\sigma_1}{\partial a}\cos(\theta + \phi) - \frac{A_1}{\omega^2}\frac{\partial A_1}{\partial a}\sin(\theta + \phi)$$

$$- \frac{1}{\omega^2}\frac{\partial f_1}{\partial q'}(q, \omega q')\Big| \left[\frac{\partial Z_1}{\partial\theta} + \frac{\sigma_1 a}{\omega}\cos(\theta + \phi) + \frac{A_1}{\omega}\sin(\theta + \phi)\right]$$

$$- \frac{1}{\omega^2}\frac{\partial f_1}{\partial q}(q, \omega q')\Big| Z_1 - \frac{1}{\omega^2} f_2\left[a\sin(\theta + \phi), a\omega\cos(\theta + \phi)\right].$$

To determine the second approximation to the solution (a, ϕ), it is necessary that Z_2 be periodic. Only the terms of period 2π on the right hand side of (36) are needed to obtain a second approximation to the subharmonic amplitude, a, and phase, ϕ, and a first approximation (35) to the harmonic terms.

For convenience the last two terms of the right side of equation (36) are expressed as the finite Fourier sum $\sum_1^r M_k(a, \phi)\sin k(\theta + \phi) + N_k(a, \phi)\cos k(\theta + \phi)$.

Then equation (36) becomes

$$\frac{\partial^2 Z_2}{\partial\theta^2} + Z_2^2 = \left(\frac{2\sigma_2}{\omega} + \frac{\sigma_1^2}{\omega^2} a - \frac{A_1}{\omega^2}\frac{\partial A_1}{\partial a}\right)\sin(\theta + \phi) \tag{37}$$

$$- \left(\frac{2A_2}{\omega} + \frac{2\sigma_1 A_1}{\omega^2} + \frac{\sigma_1 a}{\omega^2} + \frac{\partial\sigma_1}{\partial a}\right)\cos(\theta + \phi) - \frac{2}{\omega^3}(\sigma_1 + A_1)$$

$$\times \sum_2^m \left[\frac{m^2}{1 - m^2} F_m(a)\sin m(\theta + \phi) + \frac{m^2}{1 - m^2} G_m(a)\cos m(\theta + \phi)\right]$$

$$- \frac{1}{\omega^2}\sum_1^r \left[M_k(a, \phi)\sin k(\theta + \phi) + N_k(a, \phi)\cos k(\theta + \phi)\right] .$$

Thus if Z_2 is to be periodic, it is necessary that

$$\left.\begin{aligned} \sigma_2 &= -\frac{\sigma_1^{\,2}}{2\omega}\,a + \frac{A_1}{2\omega}\frac{\partial A_1}{\partial a} + \frac{1}{2\omega}M_1(a,\ \phi) \\ A_2 &= \frac{\sigma_1 A_1}{2} - \frac{\sigma_1 a}{2\omega}\frac{\partial \sigma_1}{\partial a} + \frac{N_1}{2}(a,\ \phi)\ , \end{aligned}\right\} \qquad (38)$$

where A_1 and σ_1 are given by equations (34). Equations (34) and (38) define σ_1, σ_2, A_1 and A_2 in (27). From the second and third equations of (27), the amplitude and phase of the subharmonic solution, to an approximation of the order of ϵ^2, are

$$\left.\begin{aligned} \frac{da}{dt} &= \omega\frac{da}{d\theta} = \epsilon A_1(a) + \epsilon^2 A_2(a,\ \phi) \\ \frac{d\phi}{dt} &= \omega\frac{d\phi}{d\theta} = \epsilon\sigma_1(a) + \epsilon^2\sigma_2(a,\ \phi)\ . \end{aligned}\right\} \qquad (39)$$

Now if there exist values of a and ϕ such that $\dot{a} = \dot{\phi} = 0$ and $J\left(\frac{\dot{a},\dot{\phi}}{a,\phi}\right) \neq 0$ where σ_1, σ_2, A_1 and A_2 are given by (34) and (38), then the subharmonic solution of order n exists. The complete solution to an order of ϵ is

$$x = Z_o(\theta,\ \phi,\ a) + \epsilon Z_1(\theta,\ \phi,\ a)$$

where (a, ϕ) are found from (39) and Z_o, Z_1 are given by (32) and (35). The above equations yield a method by which a second approximation to the subharmonic solution of (17) can be computed, if a periodic solution and the existence conditions derived from (39) are satisfied.

The above method of successive approximation is readily extendable to systems of several degrees of freedom provided such systems are loosely coupled and are almost linear and almost lossless. This method of generalization to several, say ℓ, degrees of freedom consists of reducing the problem to a set of ℓ second order differential equations. Bothwell[43] and

Bulgakov[51] have discussed a linear transformation method of reduction to normal form such as equation (28). Under the restrictions that the system is (a) nearly linear, (b) nearly conservative, and (c) nearly autonomous, the equations which describe a general system of ℓ degrees of freedom are

$$\sum_{j=1}^{\ell} \left(a_{ij} \frac{d^2 y_j}{dt^2} - b_{ij}\, y_j\right) + \epsilon\, g_i = 0\ , \qquad i = 1, 2, \text{---}\ n\ .$$

The quantities g_i are functions of the parameter ϵ, the independent variable t, and the dependent variables y_j and their derivatives. By means of the linear transformation

$$y_j = \sum_{k=1}^{\ell} C_{jk}\, x_k\ , \qquad j = 1, 2\ \text{---}\ n\ , \tag{40}$$

the general system of equations is transformed to

$$\frac{d^2 x_i}{dt^2} + \Omega_i^{\,2}\, x_i + \epsilon f_i = 0\ , \qquad i = 1, 2, \text{---}\ n\ . \tag{41}$$

The constants Ω_i are the natural angular resonant frequencies of the system. Thus by means of such transformations as (40) a system of ℓ degrees of freedom is reducible to ℓ quasilinear second order differential equations in which coupling terms are of order ϵ. The approximation methods given above can be applied to each of these differential equations.

As an example, consider a system of two degrees of freedom which, if one natural frequency is close to that of the excitation, is reducible to the pair of quasilinear equations

$$\frac{d^2 x_1}{dt^2} + \omega^2\, x_1 = \epsilon V_1 \cos\ \omega t - \epsilon f\ (x_1, \dot{x}_1, x_2, \dot{x}_2) \tag{42}$$

$$\frac{d^2x_2}{dt^2} + \frac{\omega^2}{n^2} x_2 = \epsilon V_2 \cos \omega t - \epsilon f_2(x_1, \dot{x}_1, x_2, \dot{x}_2) \quad .$$

This set of equations **may** possess a solution, which by (23) is

$$x_1 = X_1 \sin(\omega t + \phi_1) = X_1 \sin\Psi_1$$

$$x_2 = X_2 \sin\left(\frac{\omega t}{n} + \phi_2\right) = X_2 \sin\Psi_2 \quad .$$

X_1, X_2, ϕ_1 and ϕ_2 are determined from the equilibrium points of

$$\dot{X}_1 = \frac{\epsilon}{2\pi\omega} \int_0^{2\pi} \Big[V_1 \cos \omega t - f_1(X_1 \sin\Psi_1, \omega X_1 \cos\Psi_1 \;, \qquad (43)$$

$$X_2 \sin\Psi_2, \frac{\omega}{n} X_2 \cos\Psi_2) \Big] \cos(\omega t + \phi) d(\omega t),$$

$$\dot{\phi}_1 = -\frac{\epsilon}{2\pi X_1 \omega} \int_0^{2\pi} f_1(x_1, \dot{x}_1, x_2, \dot{x}_2) \sin(\omega t + \phi_1) d(\omega t),$$

$$\dot{X}_2 \quad \frac{\epsilon n}{2\pi\omega} \int_0^{2\pi} \Big[V_2 \cos \omega t - f_2(x_1, x_2, \dot{x}_1, \dot{x}_2) \Big] \cos\left(\frac{\omega t}{n} + \phi_2\right) d\left(\frac{\omega t}{n}\right) ,$$

and

$$\dot{\phi}_2 = -\frac{\epsilon n}{2\pi\omega X_2} \int_0^{2\pi} f_2(x_1, \dot{x}_1, x_2, \dot{x}_2) \sin \left(\frac{\omega t}{n} + \phi_2\right) d\left(\frac{\omega t}{n}\right) .$$

Higher approximations could be constructed as was done for a single degree of freedom. These higher approximations are usually unnecessary if one of the natural resonant frequencies is close to the excitation frequency.

It has been shown how the existence conditions of a subharmonic could be determined and how successive approximations to this subharmonic solution could be calculated. There remains the question of determining whether or not such solutions will be stable.

The approximation methods of Kryloff lend themselves easily to investigations of stability by the methods of Liapounoff[50]. The Kryloff approximation technique leads to pairs of equations of the form

$$\dot{a} = P(a, \phi)$$
$$\dot{\phi} = S(a, \phi) \tag{44}$$

whose equilibrium points $\dot{a} = \dot{\phi} = 0$ define a_o, ϕ_o to a certain approximation in ϵ. P and S are analytic so if (a, ϕ) are replaced by $(a_o + \alpha, \phi_o + \eta)$, where α, η are small, the left sides of (44) can be expanded in a Taylors series about a_o, ϕ_o. Since a_o, ϕ_o are solutions of the equilibrium conditions obtained from (44), it follows that only terms containing the perturbations α, η will not be equal to zero.

If only terms of the first order are retained, equations (44) are replaceable by the variational equations

$$\frac{d\alpha}{dt} = \left.\frac{\partial P}{\partial a}\right|_{a_o, \phi_o} \alpha + \left.\frac{\partial P}{\partial \phi}\right|_{a_o, \phi_o} \eta$$
$$\frac{d\eta}{dt} = \left.\frac{\partial S}{\partial a}\right|_{a_o, \phi_o} \alpha + \left.\frac{\partial S}{\partial \phi}\right|_{a_o, \phi_o} \eta \quad . \tag{45}$$

Equations (45) are a pair of first order linear differential equations and possess solutions of the form ϵ^{kt}. By Liapounoff's Stability Theorem[42], the sufficient conditions that the solution a_o, ϕ_o be stable are that the real parts of the roots of

$$\begin{vmatrix} \left(-p + \frac{\partial P}{\partial a}\right) & \frac{\partial P}{\partial \phi} \\ \frac{\partial S}{\partial a} & \left(-p + \frac{\partial S}{\partial \phi}\right) \end{vmatrix} = 0 \tag{46}$$

be negative, and that

$$J\left(\frac{P,\ S}{a,\ \phi}\right) \neq 0 .$$

If the roots have positive real parts, the solution is unstable. If the real parts of the roots p_1, p_2 of (46) are zero, the method fails and the influence of terms in η^2, $\eta\alpha$, and α^2 must be considered. Additional criteria for this case are given by Liapounoff and also Bothwell[53] and Malkin[54]. These criteria are not necessary in this thesis.

It is emphasized that one of the strong advantages of the Kryloff approximation methods is the ease with which the stability of solutions can be determined. As has been shown, the method is capable of yielding higher approximations and can be applied to systems of several degrees of freedom. A method of finding a second approximation for a system of two degrees of freedom is outlined in the Appendix.

Equivalent Linearization.

This section will treat a method of analysis in which nonlinear elements are replaced by equivalent linear elements, whose values are functions of the amplitude and relative phase of the subharmonic solution. Solutions derived by the method of equivalent linearization, as developed by Kryloff and Bogoliuboff, are shown in the books by Minorsky[33] and McLachlan[36] to satisfy the quasilinear differential equation to a first approximation in the small parameter. It will be shown later in this section that the first approximation equations can be derived in terms of the equivalent linear elements.

In the calculation of the equivalent linear elements the work of Schaffner[55] is followed. Consider a circuit with one resonant frequency and including a capacitor with small nonlinearity. The forcing function is assumed to be of the form

$$\epsilon\, V \sin n\omega t \,.$$

Let the capacitance as a function of charge be given by

$$\frac{1}{C} = \frac{1}{C_o}\left[1 + D(q)\right] \,. \tag{47}$$

It is assumed that the charge is approximated by $q = a\sin(\omega t + \phi) = a \sin\psi$, where a and ϕ are as yet undetermined. The current then is given by approximately

$$i = \omega a \cos(\omega t + \phi) \,.$$

Now if the approximate form of the charge is substituted into the equation for the nonlinear capacitance, an equivalent resistance and capacitance can be defined in terms of the components of capacitor voltage which are in phase or in quadrature with the current. The capacitor voltage is

$$v_c = \frac{q}{C_o}\,(1\ D(q)) \doteq \frac{a}{C_o}\sin\psi + \frac{a}{C_o}\sin\psi D(a\sin\psi) = \frac{a\sin\psi}{C_o} \tag{49}$$

$$+ \frac{F(a,\ \phi)}{C_o}\sin\psi + \frac{G(a,\ \phi)}{C_o}\cos\psi + \text{---} \,.$$

The equivalent linear elements are defined so as to yield a balance of real and reactive powers, W_R and W_X. That is, $W_R = \frac{1}{2}\, I_o{}^2 r_e$ and $W_X = \frac{1}{2}\, I_o{}^2 x_e$. The equivalent linear resistance is defined by

$$r_e = \frac{\text{cosine terms of } v_c \text{ at } \omega}{\omega a \cos(\omega t + \phi)} = \frac{G(a,\phi)}{\omega C_o\ a} = \frac{1}{\pi\omega a}\int_0^{2\pi} v_c \cos\psi\, d\psi \,, \tag{50}$$

and the equivalent linear reactance is defined as

$$x_e = -\frac{F(a, \phi)}{\omega C_o a} - \frac{1}{\omega C_o} = -\frac{1}{\pi \omega a}\int_0^{2\pi} v_c \sin\psi \, d\psi \ . \tag{51}$$

The negative sign is used to follow the linear circuit convention of negative capacitive reactance. The equivalent r_e, x_e when added to the physical linear impedances ωL and R, where L and R are the circuit linear inductance and resistance, define the equivalent linear circuit consisting of a series combination of these elements. That is, the quasilinear circuit has been replaced by an equivalent circuit with linear elements as shown in Figure 1. Thus the total circuit resistance is

$$r = r_e + R \ , \tag{52A}$$

and the circuit reactance is

$$x = \omega L + x_e \ . \tag{52B}$$

In order that a stable subharmonic solution exist it is necessary that the system (52) possess a set of values (a, ϕ) such that $r = x = 0$. The values of a and ϕ that satisfy the equilibrium conditions

$$\begin{aligned} 0 &= R + r_e \\ 0 &= \omega L + x_e \end{aligned} \tag{53}$$

are those that yield an approximate subharmonic solution of the circuit. If $n = 1$, the condition of principal resonance, the current in the capacitive brance of Figure 1 is given by

$$i = \frac{I\left[R^2 + (\omega L)^2\right]^{1/2}}{\left[(R + r_e)^2 + (\omega L + x_e)^2\right]^{1/2}} \sin(\omega t + \beta) \ ,$$

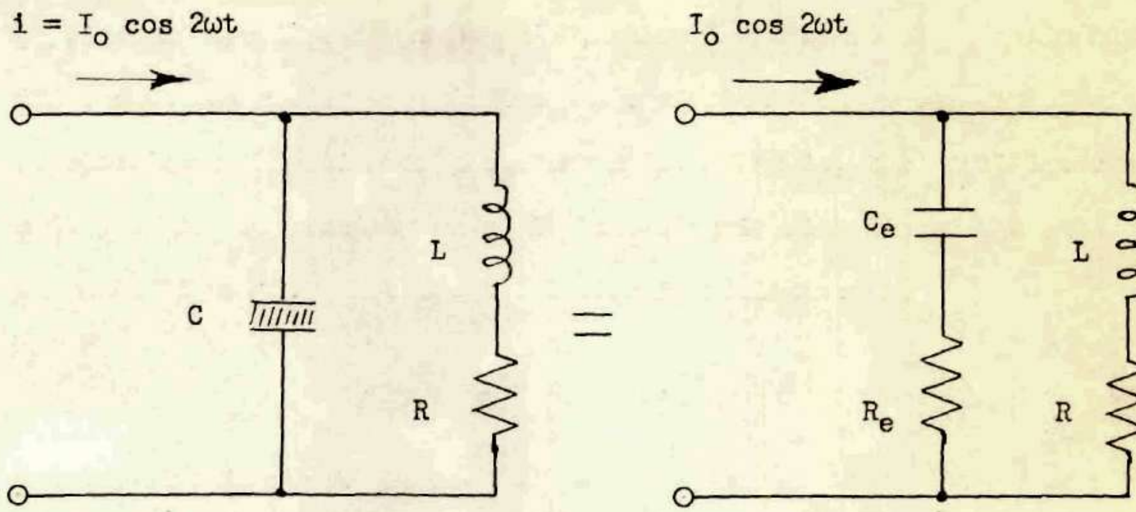


FIG. 1. EQUIVALENT LINEAR CIRCUIT.

or since the circuit losses were assumed small ($\omega L >> R$),

$$i \doteq \frac{I\omega L}{\left[(R + r_e)^2 + (\omega L + x_e)^2\right]^{1/2}} \sin(\omega t + \beta) \,, \tag{54}$$

where

$$\beta \doteq - \tan^{-1} \left(\frac{\omega L + x_e}{R + r_e}\right) \quad .$$

It was originally assumed that the current amplitude and phase were ωa and ϕ respectively. Thus the values of a and ϕ for the principal resonance solution are determined by equating (48) to the current amplitude and phase as defined by (54) from the linearized equivalent circuit. Thus a and ϕ are

$$a \doteq I \frac{\omega L}{\left[(R + r_e)^2 + (\omega L + x_e)^2\right]^{1/2}}, \phi \doteq - \tan^{-1}\left(\frac{\omega L + x_e}{R + r_e}\right), \tag{55}$$

where r_e and x_e are given by (50) and (51) with $n = 1$.

The procedure outlined above indicates how a subharmonic or principal resonance solution of a quasilinear circuit can be calculated by the method of equivalent linearization. The above procedure can be justified by returning to equation (49). In (49), it is seen that

$$F(a, \phi) = \frac{1}{\pi} \int_0^{2\pi} D(a \sin \psi) \sin^2 \psi \, d \psi \tag{56}$$

and

$$G(a, \phi) = \frac{1}{\pi} \int_0^{2\pi} D(a \sin \psi) \cos \psi \sin \psi \, d \psi \,. \tag{57}$$

If the loss, nonlinearity, detuning and driving force are of order ϵ, the quasilinear differential equation of the circuit of Figure 1(a) is given by

$$\frac{d^2q}{dt^2} + \frac{1}{C_o} q = I \cos n\omega t - R \frac{dq}{dt} - \frac{D(q)}{C_o} q \ .$$

This can be rewritten as

$$\frac{d^2q}{dt^2} + \omega^2 q = \frac{I}{L} \cos n\omega t - \frac{R}{L} \frac{dq}{dt} - \frac{D(q)}{LC_o} q - \left(\frac{1}{LC_o} - \omega^2\right) q \ , \tag{58}$$

which is in the form of equation (17). The first approximation analysis can be applied directly to equation (58). Under the assumptions made concerning loss, nonlinearity and detuning, comparing (17) and (58) it is seen that

$$\epsilon f(q, \dot{q}) = \frac{R}{L} \dot{q} + \frac{D(q)}{LC_o} q + \left(\frac{1}{LC_o} - \omega^2\right) q \ .$$

Thus the first approximation solution is given from equations (23) by

$$\left.\begin{aligned} \dot{a} &= -\frac{1}{\pi\omega}\left[\pi\omega \frac{R}{L} a + \frac{1}{LC_o}\int_0^{2\pi} D(a \sin\psi)\sin\psi \cos\psi \, d\psi \right. \\ &\qquad\qquad \left. - \frac{I}{L}\int_0^{2\pi} \cos n\omega t \cos\psi \, d\psi\right] \\ \dot{\phi} &= \frac{1}{\pi a\omega}\left[\pi a \left(\frac{1}{LC_o} - \omega^2\right) + \frac{1}{LC_o}\int_0^{2\pi} D(a \sin\psi)\sin^2\psi \, d\psi \right. \\ &\qquad\qquad \left. - \frac{I}{L}\int_0^{2\pi} \cos n\omega t \sin\psi \, d\psi\right] \ . \end{aligned}\right\} \tag{59}$$

Now if (56), (57), (50) and (51) are substituted into (59) and if $n \neq 1$, there result the equations

$$\left.\begin{aligned}\dot{a} &= -\frac{1}{L}(R + r_e)a \\ \dot{\phi} &= \frac{1}{\omega}\left(\frac{1}{LC_o} - \omega^2\right) - \frac{1}{L}\left(x_e + \frac{1}{\omega C_o}\right) = -\frac{1}{L}\left(x_e + \omega L\right)\ .\end{aligned}\right\} \quad (60)$$

The first of these is recognized as the relation for the damping of the equivalent linear circuit. If $n = 1$, the equations for this principal resonance case become

$$\left.\begin{aligned}\dot{a} &= -\frac{1}{L}(R + r_e)a + \frac{I}{\omega L}\cos\phi \\ \dot{\phi} &= -\frac{1}{L}(x_e + \omega L) - \frac{I}{\omega L}\sin\phi\ .\end{aligned}\right\} \quad (61)$$

The two pairs of equations (60) and (61) now define relations between the first approximation methods and the method of equivalent linearization. These relations show that equivalent linearization yields a solution, which is valid to a first approximation in ϵ.

A generalization of the method of equivalent linearization for several degrees of freedom consists of merely defining equivalent resistances and reactances for each degree of freedom. In general each linearized element will be a function of the amplitude and phase of each frequency component of the response.

The stability of solutions calculated by equivalent linearization for simple cases is determined by examination of the condition under which the circuit equivalent resistance goes from positive to negative. In circuits of several degrees of freedom, it is more convenient to use the equations of the first approximations and consider the stability of perturbations of their equilibrium points.

CHAPTER III

FERROELECTRIC MATERIALS AS NONLINEAR REACTIVE ELEMENTS

A ferroelectric dielectric possesses a polarization, which is a function of electric field strength. If the charge on a ferroelectric capacitor is plotted against capacitor voltage, a hysteresis loop results. Studies of the crystalline structure of ferroelectrics by von Hippel[23] and Jaynes[53] have shown that these dielectrics have different structural forms above and below the temperature of the Curie point. In pure materials, ferroelectric and hysteresis effects disappear at temperatures above the Curie point.

The dielectrics used in the experimental work for this thesis were ceramic compositions of barium titanate and strontium oxide. Several samples of commercial ferroelectric capacitors and dielectrics were obtained from manufacturers. The three dielectrics which exhibited the greatest nonlinearity were selected for use in the experimental work. Those selected were the ET61 and ET46 dielectrics of the D. M. Steward Manufacturing Company, and the K3300 dielectric of Glenco Corporation.

Since the polarization of the dielectric is a function of electric field strength, the greatest change in capacitance with voltage is obtained by the use of thin dielectrics. A lower limit to the thinness of the dielectric is set by ease of handling, the power to be dissipated in the dielectric, and the least capacity of interest. The samples used were of 10 and 20 mil thicknesses, which could be handled easily. A careful search of the literature and of manufacturers' data has failed to reveal any suitable ferroelectric materials with relative dielectric constants less than about 1000 when unbiased. The ceramic form of barium titanate has a dielectric constant of about 1500 at room temperature. These high dielectric constants coupled with the thinness of the dielectric place a limitation on the usable plate area and set a minimum capacitance. The least usable capacity attained was 300 mmf for zero bias at room temperature. This capacitor contained the ET46 dielectric of 20 mil thickness and a dielectric constant of roughly 5000 without bias.

A major difficulty is encountered in establishing the nonlinear characteristic of a dielectric and in obtaining an analytical approximation for this characteristic. Accurate measurement of the nonlinear characteristics is complicated by the temperature dependence of the dielectric constant. It is desired that the charge as a function of voltage be determined with and without bias and that an approximate relation of capacitance to charge be deduced for use in analytical work. Furthermore the subharmonic current of a circuit depends on the coefficients of the approximate nonlinear relation and these coefficients need to be accurately determined.

Three methods of capacity measurement were used in this thesis. These are (1) point by point Q meter measurements of incremental capacity as a function of applied d-c voltage, (2) dynamic hysteresis loops, (3) harmonic waveform analysis of charge through or voltage across the capacitor when a sine wave of voltage or charge is applied.

Incremental capacity, or that capacity presented to a small alternating voltage, can be measured with either a Q meter or a bridge. A circuit for measurement with a Q meter is shown in Figure 2. In this circuit C is a large paper capacitor with a capacitance much greater than the

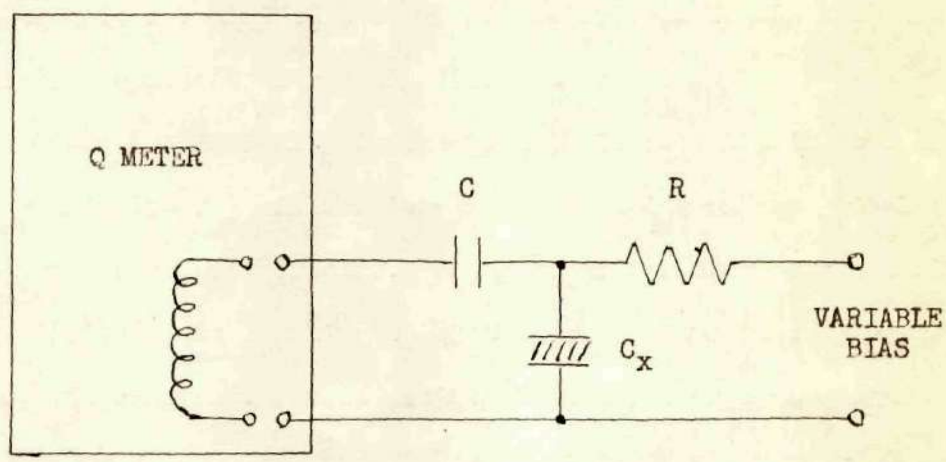


FIG. 2. INCREMENTAL CAPACITY MEASUREMENT WITH A Q METER.

sample C_x and R is a high resistance. This incremental capacity is the capacity of interest in applications in which a small alternating voltage is superimposed on a d-c bias voltage, as in certain dielectric amplifiers and dielectric modulators. Usually incremental capacitance is less than the slope of the curve of charge plotted against voltage, due to hysteresis effects. Curves of incremental capacity as a function of bias usually show a maximum capacity at a small bias and a decreasing capacity at larger biases, much as is observed in the permeability variations of ferromagnetic materials. Curves of incremental capacitance and Q as functions of bias voltage are given in Figures 3 and 4.

Dynamic plots of instantaneous charge and applied voltage can be displayed on an oscilloscope. This method was used by de Bretteville[56] to obtain hysteresis curves of $BaTiO_3$ capacitors at various temperatures. A circuit for these tests is shown in Figure 5. The capacitance C is much greater than C_x, so that almost the full applied voltage appears across C_x. The voltage across C is proportional to the integral of the current through C_x, so the voltage delivered to the vertical axis of the oscilloscope is proportional to the charge on C_x. The combination of capacitors C_2 and C_1 is a voltage divider. The unit is calibrated by observing the vertical deflection produced by a known capacitance, hence the peak deflection produced by a known charge. The calibration is thus established for a linear region of deflection of the oscilloscope. Oscilloscope pictures of the biased and unbiased hysteresis loops of charge versus voltage for the ET61 and ET46 dielectrics are shown in Figure 6. Calibration data are also given for each picture. If even order subharmonics are desired, the charge-voltage characteristic must be unsymmetrical about the bias point. Figure 6 shows that only biased dielectrics have unsymmetrical

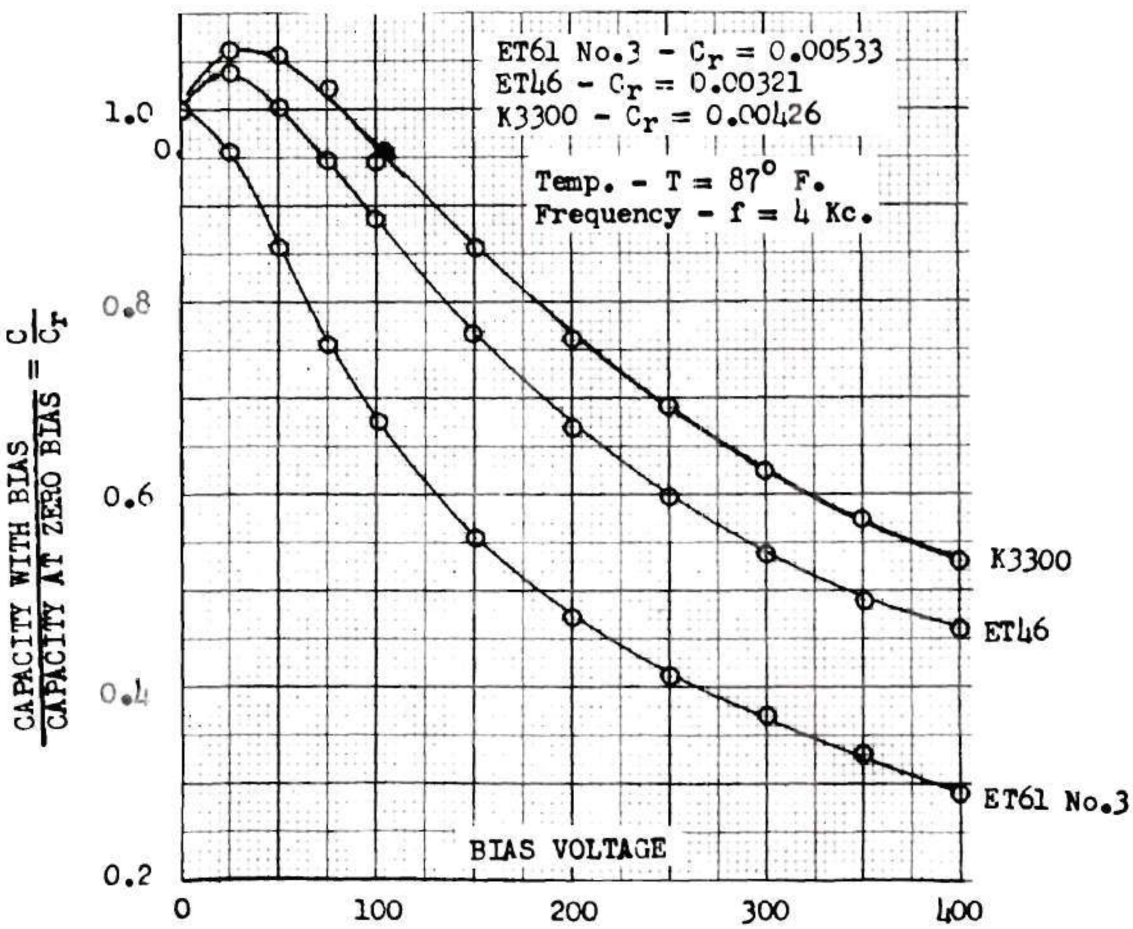


FIG. 3. INCREMENTAL CAPACITY AS A FUNCTION OF BIAS VOLTAGE

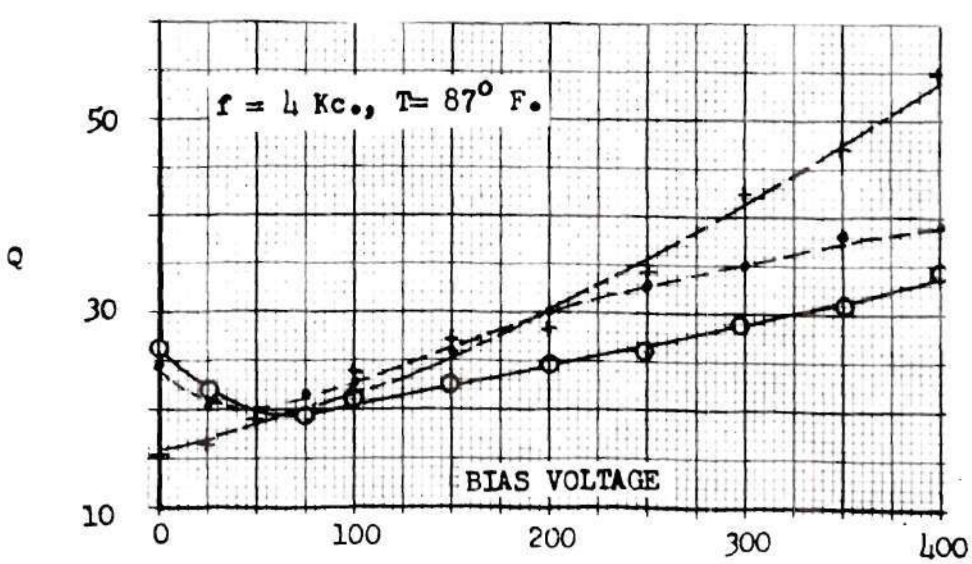


FIG. 4. CAPACITOR QUALITY FACTOR, $Q = \frac{1}{\omega C R_c}$, VERSUS BIAS VOLTAGE

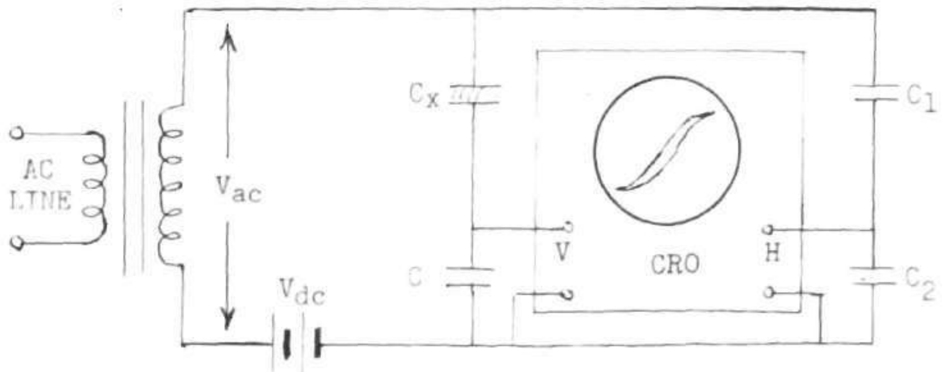
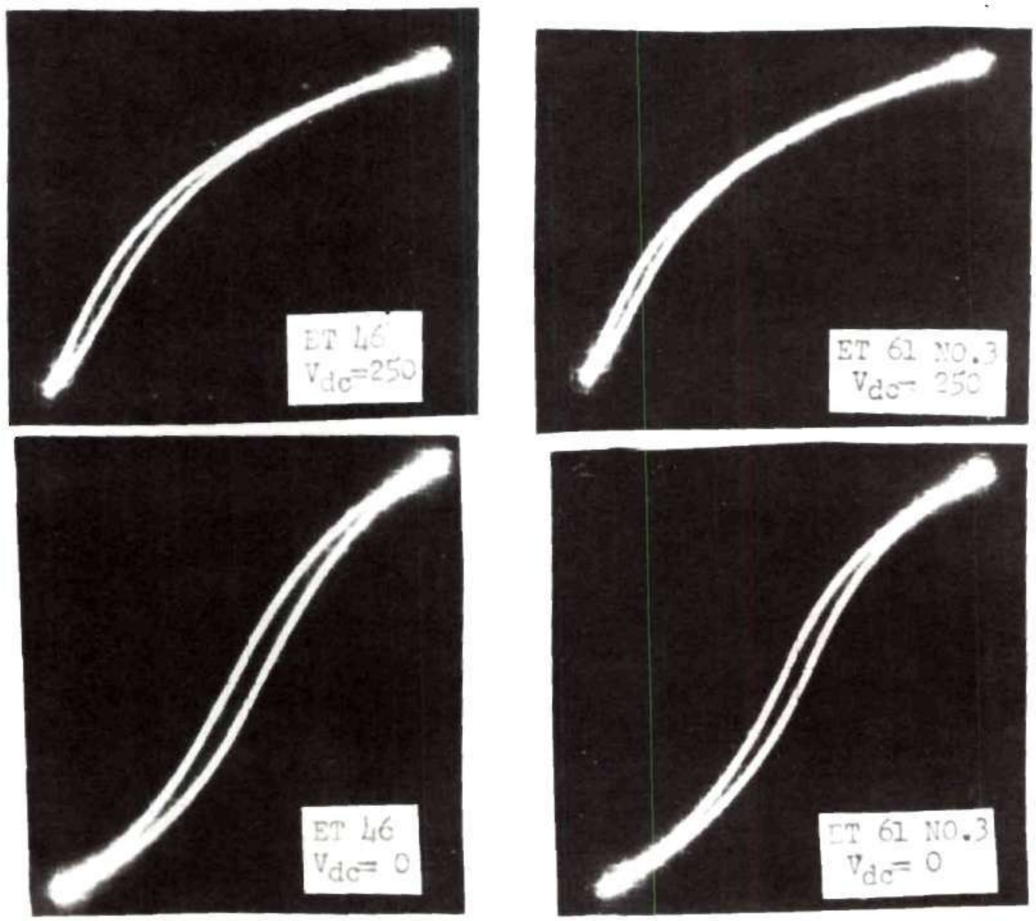


FIG. 5. TEST ARRANGEMENT FOR OBTAINING CAPACITOR HYSTERESIS LOOPS.



HOR. 1" = 310 VOLTS

T = 77° F.

VERT. 1" = 1.22 MICROCOULOMB

FIG. 6. BIASED AND UNBIASED HYSTERESIS LOOPS OF DIELECTRICS

characteristics. A polynomial approximation to such a characteristic would contain even powers of the independent variable. A characteristic, whose approximation contains terms of even powers is said to possess even order curvature.

Figure 6 shows that an appreciable degree of nonlinearity is obtained. In order that numerical calculations can be made of the operation of circuits incorporating these dielectrics it is necessary to obtain an equation to approximate their charge-voltage characteristics. The loop is of course multiple valued so that a single valued analytical approximation to it can only represent some form of average nonlinear characteristic. A polynomial is used to approximate the nonlinear characteristic. This polynomial is made to approximate the average of the upper and lower portions of the loop with large voltages. This average nonlinear characteristic is not the same as the locus of the curve traced through the tips of the hysteresis loops as the alternating voltage is varied. For small voltages the capacitance defined by such a measurement is the small signal (incremental) capacitance which is not the slope of the large signal charge-voltage characteristic. It is necessary for the large signal operation encountered in subharmonic operation to approximate the average characteristic.

Since errors in the nonlinear coefficients lead to corresponding errors in the calculated subharmonic solutions, several methods were tried in an attempt to determine more accurately the coefficients of the polynomial approximation. In one of these methods an electronic switch was used to simultaneously present a linear charge-voltage trace of a known linear capacitance and the unknown hysteresis loop. By this technique the mean slope capacity at various points of the hysteresis loop could be measured directly on the oscilloscope without having to photograph or re-trace the loop. The linear capacitance is varied to make its trace have the same slope of charge versus voltage as the mean of the hysteresis loop. It was found that errors in visual estimation of when the slopes were equal offset the advantages of this technique.

The unbiased static capacity--that is, the ratio of charge to voltage--can also be measured by the use of the electronic switch-linear capacitor arrangement, by making the linear-capacitor trace match up with the tips of the loop. This procedure would be repeated for various applied alternating voltages to obtain the capacity as a function of voltage. Since the tips of the hysteresis loops for various alternating voltages do not follow the average of a large loop, this method does not yield the large signal characteristic of interest in this thesis. The method may be useful for other applications. A curve of static capacity versus voltage, measured using the electronic switch and linear capacitor, is shown in Figure 7 for a sample of ET61 dielectric.

Figure 8 shows several charge versus voltage hysteresis loops for various applied alternating voltages with and without bias. This figure shows the variation of dielectric constant with small applied voltages. The biased loops possess a zero shift due to the presence of even harmonics.

A two-channel electronic switch was used to give simultaneous displays of the characteristics of the ET46 and ET61 dielectrics. The oscilloscope pictures of these loops shown in Figure 9 illustrate the temperature dependence of these dielectrics.

In a third method of measurement of capacitor characteristics a sine wave of voltage is applied to a series combination of the capacitor

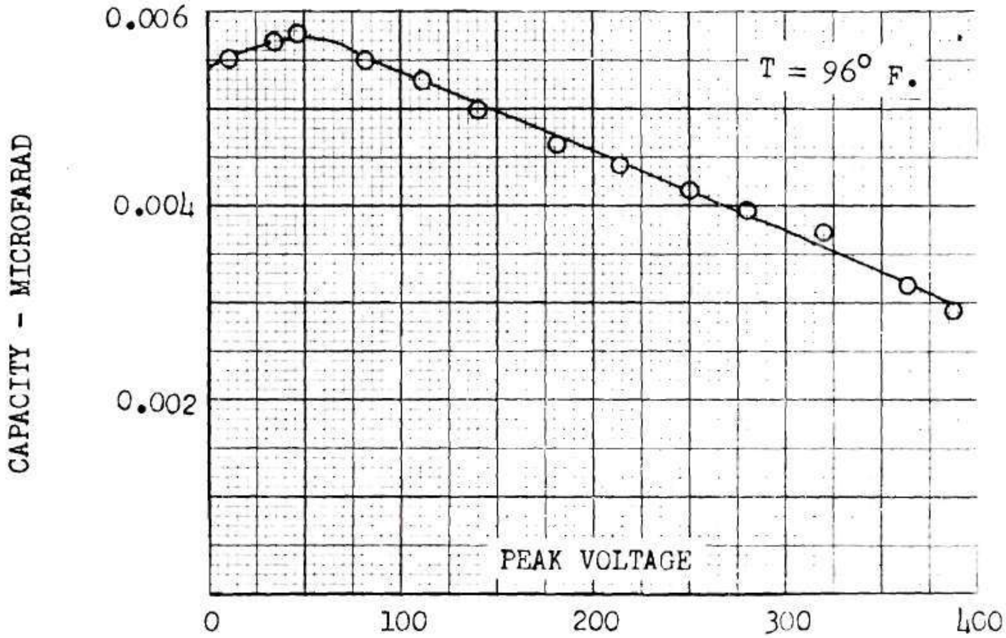
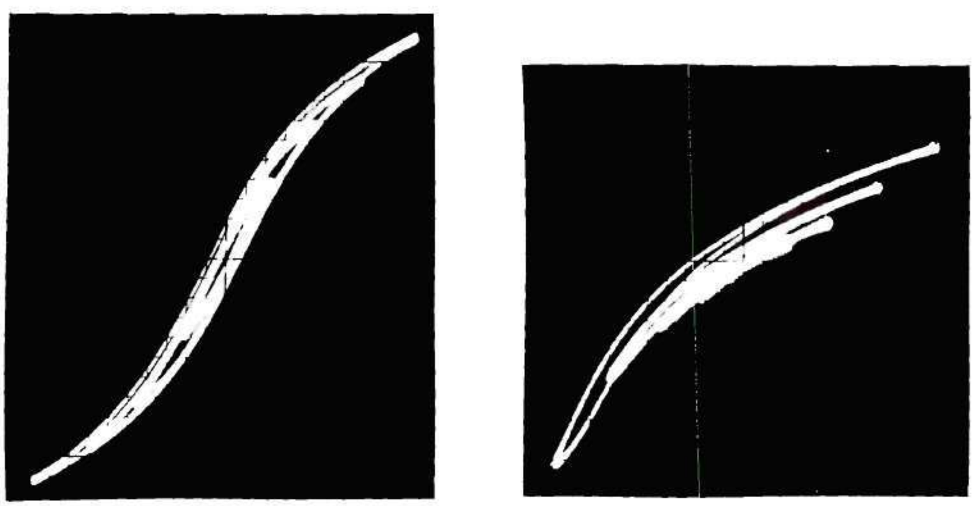


FIG. 7. STATIC CAPACITY VERSUS VOLTAGE OF ET 61 #3 DIELECTRIC.



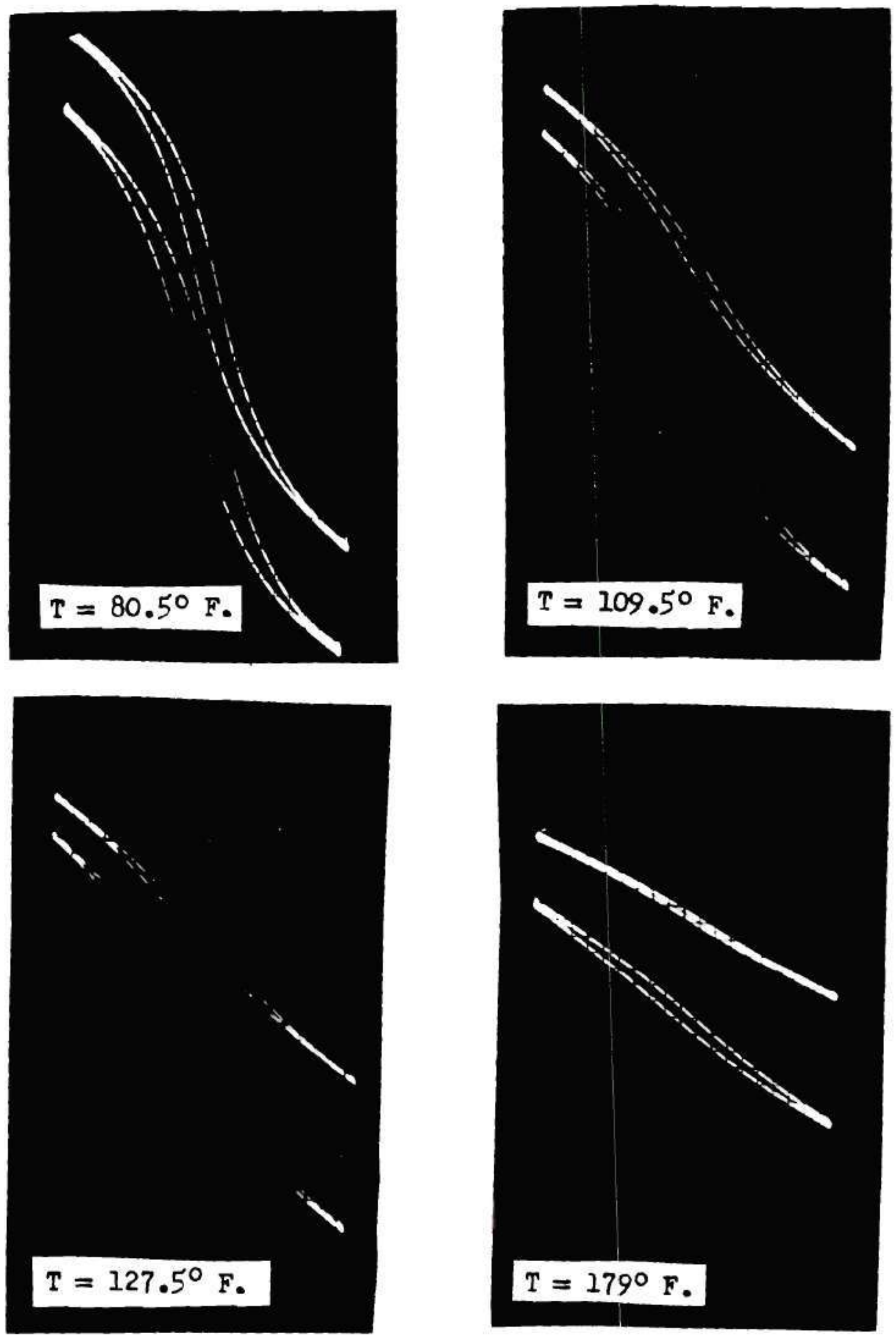
$V_{dc} = 0$ $V_{dc} = 250$

V_{ac} = 310, 216, 144, 71 PEAK V_{ac} = 304, 211, 136, 71 PEAK

ET 46 T = 77° F.

VERT. 1" = 1.22×10^{-6} COULOMB , HOR. 1" = 310 VOLTS

FIG. 8. CHARGE VERSUS VOLTAGE HYSTERSIS LOOPS



UPPER LOOP ET 61 LOWER LOOP ET 46

$V_{dc} = 0$, $V_{ac} = 310$ VOLTS PEAK

FIG. 9. VARIATION OF HYSTERSIS LOOPS WITH TEMPERATURE

under test and a large linear capacitor. At the fundamental frequency and each of its harmonics, the voltage across the linear element can be read directly and is proportional to the corresponding component of charge through the test sample. A circuit for such a measurement is shown in Figure 10. This method offers considerable promise, since the coefficients of the polynomial approximation may be computed directly from measurements of the voltage on C at harmonics of the source frequency. The sine wave alternator had less than 0.1% harmonic content, and thus insured that harmonics in the source voltage would not contribute significantly to the harmonic content of v to about the fifth harmonic for the samples used. Therefore, accurate measurements of the polynomial approximation up to the fifth power term should be possible. This method of measurement has the disadvantage that no account is taken of harmonics introduced by the multiple valued properties of the hysteresis loop; that is, no attempt is made to define a mean curvature. For this reason, harmonic analyzer measurements become less accurate for small applied voltages, since the odd harmonic content due to the finite hysteresis is a larger fraction of the total harmonic content at these frequencies. Table I shows the results of harmonic analyzer measurements of biased and unbiased K3300, ET46, and ET61 dielectrics.

Let us now consider how the polynomial approximation can be computed from the data of Table I. Roberts[57] and Urkowitz[58] report that the coefficients of this expansion decrease rapidly if the capacitor voltage is expressed as a polynomial of charge. This is also the most convenient form for solution of the differential equations which describe the circuit response. Thus, an expression for the nonlinear capacitor voltage will be sought in the form

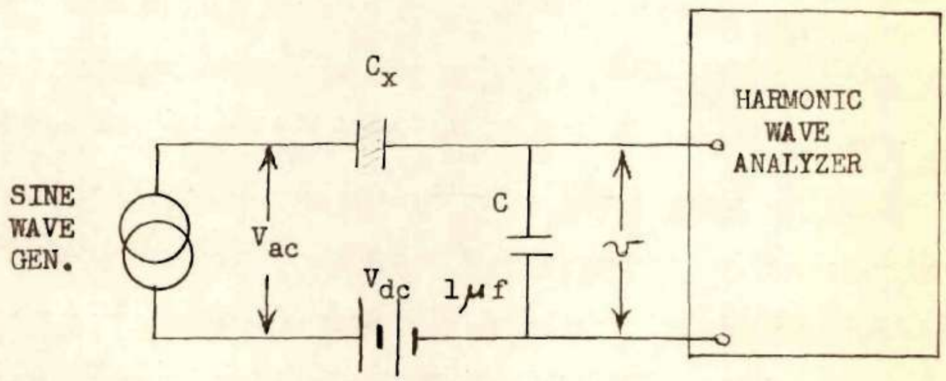


FIG. 10. HARMONIC ANALYZER MEASUREMENT OF CAPACITOR CHARACTERISTIC.

TABLE I

DATA ON HARMONIC ANALYSIS OF CHARGE AND CAPACITOR COEFFICIENTS

Sample	ET46	ET46	ET61#3	ET61#3	K3300	K3300
Temp °F	73	73	73	73	80	80
V_{dc}	0	250	0	250	0	250
V_{ac}	140	136	135	135	215	150
v at ω	0.70	0.36	0.74	0.335	1.90	0.67
v at 2ω	0.00	0.038	0.00	0.047	0.00	0.095
v at 3ω	0.051	0.0031	0.065	0.0083	0.18	0.008
v at 4ω	0.002	0.0006	0.002	0.0011	0.00	0.005
C_o uuf	6100	2720	6930	2670	11,300	4630
$a_1 \times 10^{+12}$	-----	2.7	-----	3.3	------	4.7
$a_2 \times 10^{+15}$	37	2.5	53	6.8	36.3	4.75
$D_1 \times 10^{-6}$	-----	0.37	-----	0.46	------	0.23
$D_2 \times 10^{-12}$	0.16	0.40	0.16	0.77	0.025	0.16

All voltages are r.m.s. values.

$$v_x = \frac{q}{C_x} = \frac{q}{C_o}\,(1 + D_1 q + D_2 q^2 + D_3 q^3 + - - -) \quad . \tag{62}$$

From Figure 10 the voltage, v_x, across C is

$$v_x = \frac{1}{C}\int i\,dt = \frac{q}{C} = V_{dc} + V_1 \cos(\omega t + \phi_1) + V_2 \cos(2\omega t + \phi_2)$$
$$+ \; V_3 \cos(3\omega t + \phi_3) + V_4 \cos(4\omega t + \phi_4) + V_5 \cos(5\omega t + \phi_5) + - - - .$$

The component of charge at the nth harmonic is merely V_n C, where V_n is the amplitude of the n-th harmonic capacitor voltage. If the sixth and higher harmonics are neglected, the alternating component of charge can be written as

$$q = Q_1 \cos(\omega t + \phi_1) - Q_2 \cos(2\omega t + \phi_2) - Q_3 \cos(3\omega t + \phi_3) - Q_4 \cos(4\omega t + \phi_4) - Q_5 \cos(5\omega t + \phi_5) \tag{63}$$

where

$$Q_1 = V_1\,C, \; Q_2 = V_2\,C, \; Q_3 = V_3\,C, \; Q_4 = V_4\,C, \; Q_5 = V_5\,C \; .$$

It is now assumed that the charge can be expressed in terms of the voltage v_x across C_x as

$$q = v_x(a_o - a_1 v_x - a_2 v_x^2 - a_3 v_x^3 - a_4 v_x^4) \; . \tag{64}$$

When $v_x = V \cos\omega t$ is substituted into (64), there results

$$q = a_o V \cos\omega t - \frac{a_1}{2} V^2 \cos 2\omega t - \frac{a_1}{2} V^2 - 3\frac{a_2}{4} V^3 \cos\omega t$$
$$- \frac{a_2}{4} V^3 \cos 3\omega t - \frac{a_3 V^4}{8} - \frac{a_3 V^4}{8} \cos 4\omega t - \frac{a_3 V^4}{2} \cos 2\omega t$$
$$- \frac{a_4 V^5}{16} \cos 5\omega t - \frac{5 a_4 V^5}{16} \cos 3\omega t - \frac{5 a_4 V^5}{8} \cos\omega t \; .$$

If the above equation is set equal to (63), one obtains a set of relations between Q_1 - - - Q_5 and a_o - - - a_4. If the small phase shifts introduced by the multiple valued hysteresis loop are neglected, $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$. Q_1, Q_2, Q_3, Q_4, Q_5 and V are measured quantities. The coefficients are given in terms of these quantities by

$$a_4 = \frac{16Q_5}{V^5} \qquad\qquad a_3 = \frac{8Q_4}{V^4}$$

$$a_2 = \frac{4}{V^3}(Q_3 - 5\,Q_5) \tag{65}$$

$$a_1 = \frac{2}{V^2}(Q_2 - 4\,Q_4)$$

$$a_o = \frac{1}{V}(Q_1 + 3\,Q_3 + 5\,Q_5)\quad .$$

Constant terms of the expansion of (64) are neglected, since the polynomial approximation will be expanded about the bias point. The coefficients defined by (65) yield an expression for q in terms of voltage. In order to obtain the polynomial expansion of voltage in terms of charge, it is necessary to invert (64). If (62) is substituted into (64) and terms of like powers of q equated, there results the system of equations

$$1 = \frac{a_o}{C_o} \qquad\qquad 0 = a_o\frac{D_1}{C_o} - \frac{a_1}{C_o^{\,2}}$$

$$0 = a_o\frac{D_2}{C_o} - 2\,a_1\frac{D_1}{C_o^{\,2}} - \frac{a_2}{C_o^{\,3}}$$

$$0 = a_o\frac{D_3}{C_o} - 3\,a_2\frac{D_1}{C_o^{\,3}} - \frac{a_3}{C_o^{\,4}} - \frac{a_1}{C_o^{\,2}}(D_1^{\,2} + 2\,D_2)\quad ,$$

and

$$0 = a_o \frac{D_4}{C_o} - 2\frac{a_1}{C_o^2}(D_3 + D_1 D_2) - 4\frac{D_1 a_3}{C_o^4} - \frac{a_4}{C_o^5} - \frac{3a_2}{C_o^3}(D_1^2 + D_2).$$

These equations when solved for the coefficients C_o, D_1, D_2, D_3 and D_4 yield

$$C_o = a_o \qquad\qquad D_1 = \frac{a_1}{C_o^2} \tag{66}$$

$$D_2 = 2\frac{a_1^2}{C_o^4} + \frac{a_2}{C_o^3}$$

$$D_3 = \frac{5a_1^3}{C_o^6} + \frac{5a_1 a_2}{C_o^5} + \frac{a_3}{C_o^4}$$

$$D_4 = \frac{14a_1^4}{C_o^8} + \frac{21a_1^2 a_2}{C_o^7} + \frac{3a_2^2}{C_o^6} + \frac{6a_1 a_3}{C_o^6} + \frac{a_4}{C_o^5}$$

The equations (65) and (66) when combined yield the coefficients of a polynomial approximation to the nonlinear capacitor characteristic in terms of the measured harmonic content. Table I gives the polynomial coefficients computed from equations (65) and (66).

The coefficients of the polynomial approximation of voltage in terms of charge decrease slowly for biased dielectrics. This is a result of the coefficient a_1, which is a measure of the second order curvature, not being zero for biased dielectrics. Since a_1 appears in the equation for each coefficient D_1, D_2, D_3 of equation (66), polynomial coefficients of the voltage versus charge characteristic decrease slowly unless a_1 is very small. Any errors present in the measurements are greatly increased in the inversion of the series to find the coefficients C_o, D_1, D_2, D_3 and D_4, since the errors in the a_1, a_2, a_3 and a_4 coefficients are cumulative in the calculated D_1, D_2, D_3 and D_4 values. Thus the accuracy of the coefficients of the approximation of voltage as a polynomial of charge computed from harmonic analyzer measurements with a sine wave of voltage is poor if the charge voltage characteristic is dissymmetrical.

The expression for capacitor voltage as a function of charge is much more convenient for analytical use than charge in terms of voltage. Therefore, this form of the expansion is retained, and a new measurement method sought. The harmonic measurement procedure can be reversed and the harmonic content of capacitor voltage measured when an approximate sine wave of current flows through the capacitor. Then the nonlinear coefficients D_1, D_2 etc. can be calculated directly in terms of the measured quantities without the inversion of a series.

A capacitor current, which is approximately a sinusoid, can be obtained by placing the nonlinear capacitor in a high Q resonant circuit which is driven by a high plate resistance pentode tube. Since the tube plate circuit is resonant at the exciting frequency, the current through the nonlinear capacitor has negligible harmonic content. A circuit suitable for such measurements is shown in Figure 11. A one microfarad capacitor was used in series with the unknown and the alternating voltage across this capacitor was measured. This voltage reading in volts is then equal in magnitude to the charge in microcoulombs flowing through C_x. The tube operating conditions were adjusted to minimize its harmonic distortion. The large resistor R across C serves to insure that the full bias will occur across C_x. The voltage E_D is variable so that the capacitor bias can be varied with fixed tube operating conditions. The voltage v_x, at the input frequency and harmonics thereof, is measured with a harmonic

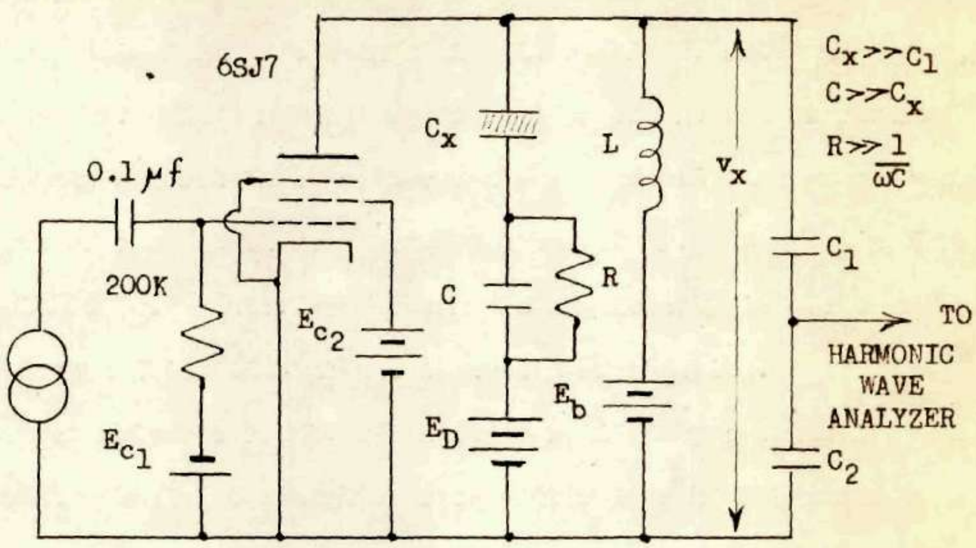


FIG. 11. HARMONIC ANALYZER MEASUREMENT OF CAPACITOR CHARACTERISTICS WITH A SINE WAVE OF CHARGE.

wave analyzer, and the voltage divider C_1, C_2.

Table 2 gives the results of measurements on several dielectric samples. Table 3 gives the coefficients of the polynomial approximation computed from these data and shows the variation of polynomial coefficients with charge. Harmonic analysis of capacitor voltage with a sine wave charge is the most accurate method found for measuring the non-linear characteristics of biased dielectrics. This method possesses the great advantage that the measurements can be made at the frequency at which the circuit is to operate, and hence that any dielectric heating will show up in the characteristic measurement.

If the capacitor charge is a sine wave of peak amplitude Q_o, and the fifth and higher harmonics are negligible, the capacitor voltage can be written as the finite Fourier sum

$$\begin{aligned} v &= V_o + V_1 \sin \omega t - V_2 \cos 2\,\omega t - V_3 \sin 3\,\omega t + V_4 \cos 4\,\omega t \\ &= \frac{q}{C_o}(1 + D_1 q + D_2 q^2 + D_3 q^3) = \frac{Q_o}{C_o} \sin \omega t + \frac{D_1 Q_o{}^2}{2\,C_o}(1 - \cos 2\,\omega t) \\ &\qquad + \frac{D_2 Q_o{}^3}{4C_o}(3 \sin \omega t - \sin 3\,\omega t) \\ &\qquad + \frac{D_3 Q_o{}^4}{8C_o}(\cos 4\,\omega t - 4 \cos 2\,\omega t + 3)\ , \end{aligned}$$

where V_o is the bias. Capacitor nonlinearity above the fourth power of q is neglected.

The harmonic amplitudes V_1, V_2, V_3, and V_4 as well as Q_o can be measured in the circuit of Figure 11. Thus the coefficients of the polynomial approximation to the capacitor characteristic, in terms of the measured quantities, are given by

TABLE 2

HARMONIC ANALYZER DATA WITH A SINE WAVE OF CHARGE

CAPACITOR BIAS 250 VOLTS, 83°F, 3.2 KILOCYCLES

Sample	q	-----Voltage Across C_x-----			
	μ coul.	V_1 at ω	V_2 at 2ω	V_3 at 3ω	V_4 at 4ω
ET61#3	0.47	192	24	2.94	0.19
ET61#3	0.385	153	17.4	1.67	0.11
ET61#3	0.27	111	9.9	.75	0.07
ET61#3	0.15	62.7	3.14	.23	0.01
ET46	0.50	191	26.7	3.74	0.40
ET46	0.415	160	18.7	2.44	0.27
K3300	0.52	128	14.2	2.54	0.67
K3300	0.30	76.1	5.35	.80	----
ET61#2	0.49	170	26.7	3.47	0.05
ET61#2	0.20	78.6	6.67	0.53	----
ET61#1	0.21	174	26.7	3.47	0.03
ET61#1	0.086	78.8	5.25	0.67	----
ET61#4	0.50	163	25.8	3.00	----
ET61#4	0.24	80	6.81	0.47	----
ET61#4	0.095	36	1.28	0.07	----

All voltages and charges in r.m.s. values

TABLE 3

COEFFICIENTS OF POLYNOMIAL APPROXIMATION FROM DATA OF TABLE 2

83°F, Bias 250 volts, 3.2 kc

Sample	Charge μ coul.	C_o $\mu\mu f$	D_1 $\times 10^{-6}$	D_2 $\times 10^{-12}$
ET61#3	0.47	2570	0.396	0.146
ET61#3	0.385	2600	0.435	0.161
ET61#3	0.27	2500	0.49	0.19
ET61#3	0.15	2420	0.48	0.32
ET46	0.50	2780	0.42	0.160
ET46	0.415	2720	0.394	0.186
K3300	0.52	4410	0.267	0.159
K3300	0.52	4410	0.267	0.159
K3300	0.30	4100	0.34	0.24
ET61#2	0.49	3080	0.48	0.182
ET61#2	0.20	2590	0.59	0.345
ET61#1	0.21	1210	1.04	0.97
ET61#1	0.086	1180	1.21	2.48
ET61#4	0.50	3260	0.475	0.157
ET61#4	0.24	3000	0.500	0.205
ET61#4	0.091	2700	0.524	0.39

$$C_o = \frac{Q_o}{V_1 - 3V_3} \tag{67}$$

$$D_1 = \frac{2}{Q_o}\frac{V_2 - 4V_4}{V_1 - 3V_3}$$

$$D_2 = \frac{4V_3}{Q_o^2(V_1 - 3V_3)}$$

$$D_3 = \frac{8V_4}{Q_o^3(V_1 - 3V_3)}$$

Table 4 gives calculated coefficients of the ET61 Sample No. 4 capacitor characteristic for various biases. This Table shows the variation of the polynomial approximation coefficients as the capacitor bias is varied. The reduction of odd order and increase of even order nonlinearity with bias is of particular interest, since the capacitor may be biased to favor an even or odd order subharmonic.

The harmonic analysis technique of measuring the polynomial coefficients necessitates some knowledge of the relative phase angles between harmonics. These are approximately multiples of 90°. These phases can usually be obtained as multiples of 90° by deduction from the knowledge that the unbiased charge-voltage characteristic has odd symmetry and that capacity decreases with voltage. These relative phases were checked by the use of a harmonic generator, which generates harmonics of variable amplitude and phase, and an oscilloscope. The fundamental frequency output was fed to the horizontal axis of the oscilloscope and the harmonic generator output, including fundamental plus harmonics, was fed the vertical axis. The amplitudes and relative phases of the harmonics can be varied until the Lissajous pattern appears the same as an observed actual

TABLE 4

POLYNOMIAL COEFFICIENT VARIATION WITH BIAS

ET61#4, 87°F, 4 kc, Q = 0.4μcoul.

Bias = $E_b - E_D$ volts	C_o μμf	D_1 x 10^{-6}	D_2 x 10^{-12}
0	5820	-----	0.480
50	5720	0.156	0.443
100	4980	0.383	0.472
150	4400	0.510	0.427
200	3330	0.521	0.380
250	3180	0.554	0.338

hysteresis loop. The relative amplitudes and phases set on the harmonic generator then yield an approximate harmonic analysis of the actual capacitor characteristic. If a harmonic generator with very pure output voltages is used with a dual beam oscilloscope, this synthesis method is capable of yielding an accurate measurement of the nonlinear characteristic. In such a measurement the actual capacitor hysteresis loop would be placed on one set of vertical and horizontal deflection plates and the synthesized pattern on the other. Thus a very close comparison of the actual and synthesized patterns could be achieved.

The coefficients of the polynomial approximations to the capacitor voltage as a function of charge characteristic will vary if the capacitor plate area is varied. This is a result of the characteristics being approximated in terms of the practical quantities voltage and charge. The dielectric itself has a characteristic which can be approximated by expressing the electric field strength E as the polynomial

$$E = k_o D + k_1 D^2 + k_2 D^3 .$$

In this equation and only here D is used to denote electric displacement. If the dielectric is homogenous and of uniform thickness and if fringing effects are neglected, the charge and voltage of a parallel plate capacitor are given by

$$Q = D A \qquad V = E d \quad ,$$

where A is the plate area and d the dielectric thickness. Then the nonlinear voltage-charge characteristic is

$$V = d \left(k_o \frac{q}{A} + k_1 \frac{q^2}{A^2} + k_2 \frac{q^3}{A^3} \right)$$

or

$$V = \frac{dk_o}{A} q \left(1 + \frac{k_1}{k_o}\frac{q}{A} + \frac{k_2}{k_o}\frac{q^2}{A^2}\right) = \frac{q}{C_o}(1 + D_1 q + D_2 q^2) .$$

Thus the coefficient C_o is proportional to the plate area. D_1 is inversely proportional to A, and D_2 is inversely proportional to A^2. So if two capacitors are composed of the same dielectric with equal thickness, the D_1 coefficient of the larger capacitor will be less than that of the smaller by the ratio of the lesser linear capacity to the greater linear capacity. Table 3, for different samples of ET61 dielectric, indicates that the coefficients D_1 and D_2 vary approximately as discussed above.

The degree of nonlinearity achieved with the above ceramic materials does not approach the maximum obtainable. Anderson[27] and Merz have shown that single $BaTiO_3$ crystals with oriented domains may have dielectric constants of the order of 100,000 when unsaturated and 300 when saturated.

No entirely adequate technique of loss measurement is known. In the experimental circuits a Q meter was used to measure the apparent Q with the capacitor mounted in the circuit. A moderate value of Q meter drive was used in these tests to cause the capacitor to operate out of the region of increasing dielectric constant and to get a better approximation to the loop area under operating conditions.

CHAPTER IV

SUBHARMONICS IN SINGLE LOOP CIRCUITS

In this chapter subharmonic responses encountered in single loop circuits or circuits of one resonant frequency are considered. The second order subharmonic will be studied in detail.

Consider the circuits shown in Figure 12. The capacitor is assumed to have a voltage-charge characteristic

$$v = \frac{q}{C_o}\,[1 + (q)] \;.$$

The reciprocal of capacitance defined as $\frac{v}{q}$, is then

$$\frac{1}{C} = \frac{1}{C_o}\,[1 + f(q)] \;.$$

In Figure 12(a)

$$i_L = i - i_c = I_o \cos(n\omega\, t - \xi) - \frac{dq}{dt}$$

and

$$v = i_L\, R_L + L\,\frac{di_L}{dt} = \frac{dq}{dt}\, R_c + \frac{q}{C_o}\,[1 + f(q)\ ].$$

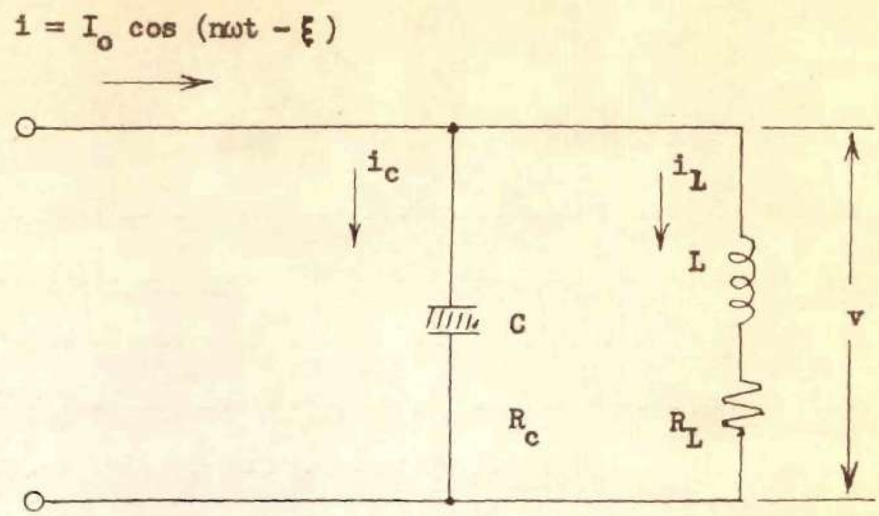
Thus,

$$R_L\left(i - \frac{dq}{dt}\right) + L\,\frac{di}{dt} - L\,\frac{d^2q}{dt^2} = R_c\,\frac{dq}{dt} + \frac{q}{C_o}\,[1 + f(q)] \;.$$

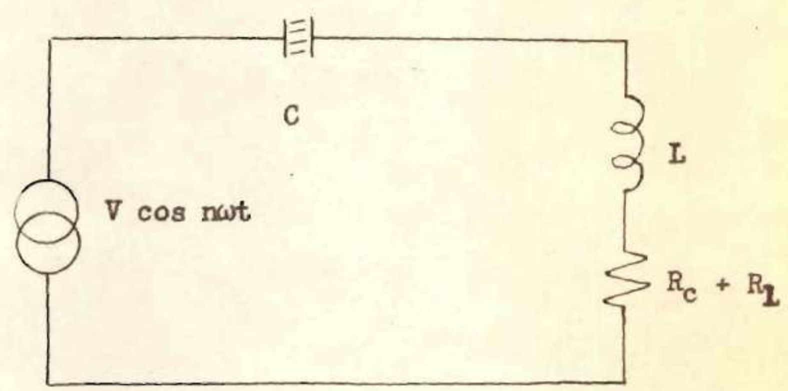
The differential equation of the capacitor charge of Figure 12(a) is then

$$L\,\frac{d^2q}{dt^2} + (R_L + R_c)\,\frac{dq}{dt} + \frac{q}{C_o}\,[1 + f(q)] = -\,n\omega L\, I_o \sin(n\omega t - \xi) \qquad (68)$$

$$+\, I_o\, R_L \cos(n\omega t - \xi) = I_o |Z| \cos n\omega t,$$



A. CURRENT DRIVEN



B. VOLTAGE DRIVEN

FIG. 12. SINGLE LOOP CIRCUITS

where

$$|Z| = \sqrt{R_L^{\,2} + n^2\,\omega^2\,L^2}$$

and

$$\xi = \cos^{-1}\frac{R_L}{|Z|} \quad .$$

For the voltage-driven circuit of Figure 12(b) the charge is the solution of

$$L\,\frac{d^2q}{dt^2} + (R_c + R_L)\,\frac{dq}{dt} + \frac{q}{C_o}\,[1 + f(q)] = E\cos n\omega t \ . \qquad (69)$$

The purposes of this chapter are to derive approximate solutions of equations (68) and (69) and to present experimental results of measurements on the circuits of Figure 12.

Second Order (Half Frequency) Subharmonics.

If a subharmonic of order two is to exist in the circuits of Figure 12, the frequency of the driving force must be such that $n = 2$ in (68) and that $\omega^2 - \frac{1}{LC}$ is small. It is further necessary that $q\,f(q)$ include even powers of q.

Let it be assumed that $n = 2$ in equation (68) and that the losses and detuning (as the difference, $\omega^2 - \frac{1}{LC}$, is sometimes called) are small. Also let

$$f(q) = D_1 q + D_2 q^2 + \text{- - - -} \ .$$

Solutions of (68) or (69) can be calculated using the analytical approximation methods of Chapter III. First, the analytical developments will be carried out and later in this section experimental results will be presented.

Perturbation solution--A solution of equation (68) is developed below from the perturbation method of Chapter II. Equation (68) can be reduced to the form of equation (1) of Chapter II by making the following substitutions.

$$D_2 = 0, \quad \Omega_o^{\,2} = \frac{1}{L\,C_o}, \; \Omega_o^{\,2}\, D_1 = \epsilon$$

$$\frac{R_L + R_c}{L} = k\,\epsilon^2, \text{ and } \frac{I_o|Z|}{L} = \epsilon\,F \quad .$$

When these quantities are substituted into equation (68) it becomes

$$\ddot{q} + \Omega_o^{\,2}\, q = \epsilon\, F \cos 2\omega t - \epsilon^2 k\, \dot{q} - \epsilon\, q^2,$$

or, in terms of the independent variable $\theta = \omega t$,

$$\omega^2 q'' + \Omega_o^{\,2}\, q = \epsilon\, F \cos 2\theta - \epsilon^{\,2}\, \omega k\, q' - \epsilon q^2 \quad , \tag{70}$$

which is of the form of equation (2). The solution can then be expressed in terms of the perturbation series

$$\begin{aligned} q &= y_o + \epsilon\, y_1 + \epsilon^2 y_2 + - - - \\ \omega &= \omega_o + \epsilon\, \omega_1 + \epsilon^2\, \omega_2 + - - - \quad , \end{aligned} \tag{71}$$

and the coefficients of this series can be calculated by the methods of Chapter II. Equation (4) shows that $\omega_o = \Omega_o$ and that the generating function to which the solution of (70) reduces as ϵ vanishes is

$$y_o = A \cos \theta + B \sin \theta \quad . \tag{72}$$

Comparison of equations (2) and (70) shows that

$$f_2(\omega q', q) = \omega k\, q'$$

and

$$f_1(\omega q', q) = q^2 .$$

Then the summation in equation (8) becomes

$$\sum_1^s \left[H_m \cos m\,\theta + K_m \sin m\,\theta\right] = \frac{A^2}{2}(1 + \cos 2\,\theta) + A\,B \sin 2\,\theta + \frac{B^2}{2}(1 - \cos 2\theta).$$

So it follows from equation (8) that for y_1 to be periodic it is necessary that $\omega_1 = 0$ and

$$y_1 = - \frac{F}{3\Omega_o^{\,2}} \cos 2\,\theta - \frac{1}{2\Omega_o^{\,2}}(A^2 + B^2) + \frac{1}{6\Omega_o^{\,2}}(A^2 - B^2)\cos 2\theta + \frac{AB}{3\Omega_o^{\,2}} \sin 2\,\theta . \tag{73}$$

Now, if these results are substituted into equation (6), this equation, which defines the solution to a second approximation, becomes

$$\Omega_o^{\,2}(y''_2 + y_2) = 2\,\Omega_o\,\omega_2(A \cos\theta + B \sin\theta) - k\,\Omega_o(-A \sin\theta + B \cos\theta) - \frac{2}{\Omega_o^{\,2}}(A\cos\theta + B\sin\theta) \times \left[\frac{A^2 - B^2 - 2F}{3} \cos 2\,\theta + \frac{AB}{3} \sin 2\,\theta - \frac{A^2 + B^2}{2}\right] . \tag{74}$$

If y_2 and the second approximation to the solution of (70) are to be periodic, the coefficients of both $\cos\theta$ and $\sin\theta$ on the right hand side of (74) must be zero. The conditions similar to (13) that y_2 be periodic are given by

$$P(A, B) = 2\,\Omega_o\,\omega_2\,A - k\,B\Omega_o - \frac{1}{\Omega_o^{\,2}}\left[\frac{A}{3}\left(\frac{A^2 - B^2}{2} - F\right) - A(A^2 + B^2) + \frac{AB^2}{3}\right] = 0 \tag{75}$$

and

$$Q(A, B) = 2\Omega_o\,\omega_2\,B + k\,A\,\Omega_o - \frac{1}{\Omega_o^{\,2}}\left[-\frac{B}{3}\left(\frac{A^2 - B^2}{2} - F\right) - B(A^2 + B^2) + \frac{A^2B}{3}\right] = 0\;. \tag{76}$$

The values of A and B which satisfy equations (75) and (76) yield the subharmonic amplitude

$$Y = \sqrt{A^2 + B^2}$$

and phase

$$\beta = \cot^{-1}\frac{A}{B}\,,$$

to an accuracy of the order of ϵ^2.

If neither A nor B is zero, the solution of equations (75) and (76) is conveniently found as follows. First (75) is divided by A and (76) by B, so that the equations to be solved simultaneously are

$$0 = 2\,\Omega_o\,\omega_2 - k\,\frac{B}{A}\,\Omega_o + \frac{1}{\Omega_o^{\,2}}\left[\frac{2A^2}{3} + B^2 + \frac{F}{3}\right]$$

and

$$0 = 2\,\Omega_o\,\omega_2 + k\,\frac{A}{B}\,\Omega_o + \frac{1}{\Omega_o^{\,2}}\left[A^2 + \frac{2}{3}\,B^2 - \frac{F}{3}\right]\;.$$

The addition of the two preceeding equations yields

$$4\,\Omega_o\,\omega_2 + k\left(\frac{A}{B} - \frac{B}{A}\right)\,\Omega_o + \frac{5}{3\Omega_o^{\,2}}\,(A^2 + B^2) = 0\;, \tag{77}$$

and subtraction of the first from the second gives

$$\Omega_o \, k \left(\frac{A}{B} + \frac{B}{A}\right) - \frac{2F}{3\Omega_o^{\,2}} = 0 \quad . \tag{78}$$

Equation (78) can be solved for $\frac{A}{B}$ by multiplying it by $\frac{A}{B}$ and using the quadratic formula. The result is

$$\frac{A}{B} = \frac{F}{3k\Omega_o^{\,3}} \pm \sqrt{\frac{F^2}{9k^2\Omega_o^{\,6}} - 1} \quad . \tag{79}$$

Since

$$\frac{A}{B} + \frac{B}{A} = \cos\beta + \tan\beta = \frac{2}{\sin 2\beta} = \frac{2F}{3k\Omega_o^{\,3}} \, ,$$

the subharmonic phase can be expressed as

$$\beta = \frac{1}{2} \sin^{-1} \frac{3k\Omega_o^{\,3}}{F} \quad . \tag{80}$$

Equation (79) yields

$$\frac{A}{B} - \frac{B}{A} = \pm\, 2 \sqrt{\frac{F^2}{9k^2\Omega_o^{\,6}} - 1} \quad ,$$

which can be substituted into (77) to obtain the subharmonic amplitude as given by

$$Y = \sqrt{A^2 + B^2} = \pm\left[\frac{6}{5} \, \Omega_o^{\,2}\left(-\, 2\Omega_o \, \omega_2 \mp \sqrt{\frac{F^2}{9\Omega_o^{\,4}} - k\Omega_o^{\,2}}\,\right)\right] . \tag{81}$$

Since $\omega_1 = 0$, equation (71) gives

$$\omega = \omega_o + \epsilon^2 \, \omega_2 + - - - - - ,$$

or

$$\omega^2 \doteq \Omega_o^{\ 2} + 2\,\epsilon^2 \Omega_o\, \omega^2$$

to a first approximation in ϵ. When this frequency relation is substituted into (81), the subharmonic amplitude and phase as given by (81) and (82) respectively can be expressed in terms of the original circuit constants as

$$Y = \text{Subharmonic Amplitude} = \pm\left\{\frac{6}{5D_1^{\ 2}\Omega_o^{\ 2}}\left[-\left(\omega^2 - \Omega_o^{\ 2}\right) \pm \sqrt{\left(\frac{I_o|Z|D_1}{3}\right)^2 - \left(\frac{R_L + R_c}{L}\right)^2 \Omega_o^{\ 2}}\right]\right\}^{1|2} \tag{82}$$

and

$$\beta = \text{Subharmonic phase} = \frac{1}{2}\sin^{-1}\frac{3(R_L + R_c)}{I_o|Z|D_1\Omega_o L}\ . \tag{83}$$

The complete solution to an order ϵ^2 including harmonic terms is given by

$$q = y_o + \epsilon\, y_1 \quad ,$$

where y_o and y_1 are given by (72) and (73) respectively. If harmonic terms of the order ϵ^2 or less in amplitude are neglected, the charge on the nonlinear capacitor of Figure 12(a), for n = 2, is

$$q \doteq Y\cos(\omega t - \beta) - \frac{I_o|Z|}{3}\cos 2\,\omega t + \frac{DY^2}{2}\left[-1 + \frac{1}{3}\cos(2\,\omega t - \beta)\right], \tag{84}$$

where Y and β are given by (82) and (83), respectively.

The perturbation method has been used above to derive second approximations for the second order subharmonic amplitude (82) and phase (83) as well as the complete solution (84). The complete solution includes terms at the forcing frequency and multiples of the subharmonic frequency.

<u>Solution by the Kryloff second approximation</u>--In the circuits of Figure 12, when the forcing function is small, it is necessary to use the second approximation. Equation (70) is reduced to the form of (28) by adding a term $(\omega^2 - \Omega_o{}^2)q$ to both sides. Hence the Kryloff second approximation solution of (70) can be calculated directly from the equations of Chapter II. By equation (32), the zeroth approximation is

$$Z_o = a \sin(\theta + \phi) = a \sin(\omega t + \phi) \ . \tag{85}$$

The detuning term, $(\omega^2 - \Omega_o{}^2)\, q = \epsilon^2\, hq$, is taken of order ϵ^2. Now since $f_1(q, \dot{q}) = q^2$, $f_2 = k\omega \frac{dq}{d\theta} - hq$ in equation (28), it follows that equations (33) and (35) become

$$\frac{\partial^2 Z_1}{\partial \theta^2} + Z_1 = \frac{F}{\omega^2} \cos 2\,\theta + 2\frac{\sigma_1}{\omega} a \sin(\theta + \phi)$$
$$- 2\frac{A_1}{\omega} \cos(\theta + \phi) - \frac{1}{\omega^2}\left[- \frac{a^2}{2} - \frac{a^2}{2} \cos(2\,\theta + 2\,\phi)\right],$$

and

$$Z_1 = - \frac{F}{3\omega^2} \cos 2\,\theta - \frac{a^2}{2\omega^2} - \frac{a^2}{6\omega^2} \cos\,(2\,\phi + 2\,\phi) \ . \tag{86}$$

Thus, as may be seen from equation (34), for Z_1 to be of period 2π, it is necessary that $\sigma_1 = A_1 = 0$.

Equation (36) then reduces to

$$\frac{\partial^2 Z_2}{\partial \theta^2} + Z_2 = \frac{a}{\omega}\left(2\sigma_2 + \frac{h}{\omega} + \frac{5a^2}{6\omega^3} - \frac{F}{3\omega^3}\cos 2\phi\right)\sin(\theta+\phi) \tag{87}$$

$$+\frac{1}{\omega}\left(-2A_2 - \frac{ak}{\omega} + \frac{Fa}{3\omega^3}\sin 2\phi\right)\cos(\theta+\phi) + \frac{a^3}{6\omega^4}\sin 3(\theta+\phi)$$

$$+ \frac{Fa}{3\omega^4}\sin(3\theta+\phi) \text{ .}$$

In order that equation (87) possess a periodic solution, it is necessary that

$$\sigma_2 = -\frac{1}{2\omega}\left(h + \frac{5a^2}{6\omega^2} - \frac{F}{3\omega^2}\cos 2\phi\right) \text{ ,}$$

and

$$A_2 = \frac{a}{2\omega}\left(-k\omega + \frac{F}{3\omega^2}\sin 2\phi\right) \text{ .} \tag{88}$$

Then by equations (39), since $\sigma_1 = A_1 = 0$, the second approximations to the amplitude, a, and phase, ϕ, of the subharmonic component of charge are solutions of

$$\frac{da}{dt} = \epsilon^2 A_2(a, \phi) = \frac{a\epsilon^2}{2\omega}\left(-k\omega + \frac{F}{3\omega^2}\sin 2\phi\right) = 0 \tag{89}$$

$$\frac{d\phi}{dt} = \epsilon^2\sigma_2(a, \phi) = -\frac{\epsilon^2}{2\omega}\left(h + \frac{5a^2}{6\omega^2} - \frac{F}{3\omega^2}\cos 2\phi\right) = 0 \tag{90}$$

Values of a and ϕ which determine a solution must correspond to an equilibrium point $\dot{a} = \dot{\phi} = 0$. By the implicit function theorem, the equations $\dot{a} = 0$ and $\dot{\phi} = 0$ possess a solution (a, ϕ) provided their Jacobian is not zero for this (a, ϕ).

Equations (89) and (90) have solutions (other than $a = 0$), if and only if,

$$F^2 \geq (3\omega^3 k)^2 \text{ and } 3\omega^2 h \geq -\sqrt{F^2 - (3\omega^3 k)^2}.$$

If $a \neq 0$, the roots of (89) and (90) are

$$\cos 2\phi = \pm\sqrt{1 - \left(\frac{3\omega^3 k}{F}\right)^2} \tag{91}$$

and

$$a^2 = \pm\frac{2F}{5}\sqrt{1 - \left(\frac{3\omega^3 k}{F}\right)^2} - \frac{6\omega^2 h}{5} \quad , \tag{92}$$

respectively. The Jacobian in question is

$$J\left(\frac{\dot a,\ \dot\phi}{a,\ \phi}\right) = \frac{\partial \dot a}{\partial a}\frac{\partial \dot\phi}{\partial \phi} - \frac{\partial \dot a}{\partial \phi}\frac{\partial \dot\phi}{\partial a} =$$

$$-\frac{\epsilon^4 F}{6\omega^4}\left(-k\omega + \frac{F}{3\omega^2}\sin 2\phi\right)\sin 2\phi + \frac{5\epsilon^4 a^2 F}{18\omega^6}\cos 2\phi.$$

At the equilibrium point, this reduces to

$$J\left(\frac{\dot a,\ \dot\phi}{a,\ \phi}\right)_{a_o,\phi_o} = \pm\frac{5\epsilon^4 F}{9\omega^6}\left\{+\frac{F}{5}\sqrt{1 - \left(\frac{3\omega^3 k}{F}\right)^2} - \frac{3\omega^3 k}{5}\right\}\sqrt{1 - \left(\frac{3\omega^3 k}{F}\right)^2} . \tag{93}$$

An examination of (92) and (93) shows that the Jacobian cannot be zero for any values of F such that a^2 is positive and real. The equilibrium conditions obtained by setting $\dot a$ equal zero in (89) and $\dot\phi$ equal zero in (90) then have the simultaneous solution given by (91) and (92). Then a subharmonic exists and its phase and amplitude are determined from equations (91) and (92).

The existence of subharmonic solutions has been proven and their amplitudes and phases calculated. In terms of the original circuit constants, equation (92) and (91) are

$$a = \pm\left\{\frac{6\omega^2}{5D_1^2\Omega_o^4}\left[-(\omega^2-\Omega_o^2)\right.\right.$$
$$\left.\left.\pm\sqrt{\left(\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}\right)^2-\frac{\omega^2}{L^2}(R_L+R_c)^2}\right]\right\}^{1|2} \quad (94)$$

and

$$2\phi = \sin^{-1}\frac{3\omega^3(R_L+R_c)}{D_1\Omega_o^2 L I_o |Z|} \quad . \quad (95)$$

It is clear from (94) that four solutions exist. The stability of these solutions is investigated below.

Let

$$a = a_o + \alpha \quad ,$$
$$2\phi = 2\phi_o + \eta \quad ,$$

where a_o and ϕ_o are given by (94), (95) and α and η are small perturbations. Variational equations are used to describe the system behavior in the vicinity of the equilibrium point (a_o, ϕ_o). If $a_o + \alpha$ and $2\phi_o + \eta$ are substituted into equations (89), (90), and only first powers of α and η retained, the variational equations are

$$\frac{da_o}{dt}+\frac{d\alpha}{dt} = \epsilon^2 A_2(a_o, \phi_o) + \frac{\alpha\epsilon^2}{2\omega}\left(-k\omega+\frac{F}{3\omega^2}\sin 2\phi_o\right)$$
$$+\frac{a_o\epsilon^2}{6\omega^3}\eta F\cos 2\phi_o ,$$

or

$$\frac{d\alpha}{dt} = \frac{\epsilon^2}{2\omega}\left(-k+\frac{F}{3\omega^2}\sin 2\phi_o\right)\alpha + \frac{a_o\epsilon^2 F}{6\omega^3}\eta\cos 2\phi_o \quad , \quad (96)$$

and

$$\frac{d\eta}{dt} = -5\,\frac{\epsilon^2 a_o}{6\omega^3}\,\alpha - \frac{F\epsilon^2}{6\omega^3}\,\eta \sin 2\,\phi_o \;. \tag{97}$$

Equations (96) and (97) are linear differential equations with constant coefficients. They possess a solution which is the sum of terms $e^{p_1 t}$ and $e^{p_2 t}$, where p_1 and p_2 are the roots of

$$\begin{vmatrix} -p + \frac{\epsilon^2}{2\omega}\left(-k\omega + \frac{F}{3\omega^2}\sin 2\phi_o\right) & a_o\,\frac{\epsilon^2 F}{6\omega^3}\cos 2\phi_o \\ -5\,\frac{\epsilon^2 a_o}{6\omega^3} & -\frac{F\epsilon^2}{6\omega^3}\sin 2\phi_o - p \end{vmatrix} = 0. \tag{98}$$

The solutions are stable if both roots are distinct and have negative real parts, or conversely, the solutions are unstable if the roots are distinct and at least one has a positive real part. As pointed out in Chapter II, if either multiple roots or zero real parts exist, the effects of higher order terms in α and η must be considered.

The expansion of the determinant (98) yields the quadratic equation,

$$P^2 + \frac{F\epsilon^2}{6\omega^3}\,P \sin 2\,\phi_o + 5\frac{\epsilon^4 F a_o^2}{36\omega^6}\cos 2\,\phi_o = 0\ ,$$

since

$$\sin 2\,\phi_o = \frac{3\omega^3 k}{F}\ .$$

The roots P_1 and P_2 of this equation are

$$P_1, P_2 = \frac{F\,\epsilon^2}{12\omega^3} \sin 2\phi_o \left[-1 \pm \sqrt{1 - \frac{20 a_o^2 \cos 2\phi_o}{F \sin^2 2\phi_o}} \right] \tag{99}$$

$$= \frac{\epsilon^2 k}{4} \left(-1 \pm \sqrt{1 - \frac{20 a_o^2 F}{9\omega^6 k^2} \cos 2\phi_o} \right)$$

$$= \frac{\epsilon^2 k}{4} \left(-1 \pm \sqrt{1 - \frac{20 a_o^2 F}{9\omega^6 k^2} \left[\pm \sqrt{1 - \frac{(3\omega^3 k)^2}{F^2}} \right]} \right).$$

Now, since $a_o^2 > 0$ and $F^2 > (3\omega^3 k)^2$, if a subharmonic is to exist, the roots P_1 and P_2 will have negative real parts for the + sign in the inner bracket. Thus the solution given by a_o and ϕ_o is stable if $\cos 2\phi_o$ is positive. The plus or minus signs on the inner radical of equation (94) occur for positive and negative values of $\cos 2\phi_o$ respectively. Then equation (94) with the positive sign on the inner radical yields the amplitude of a pair of stable subharmonics differing 180° in phase, and the negative sign on the radical yields the amplitude of a pair of unstable subharmonics.

It was pointed out earlier that equations (89) and (90) also possess an equilibrium point given by

$$a = 0 \qquad \text{and} \qquad \cos 2\phi = \frac{3\omega^2 h}{F} .$$

At this equilibrium point the subharmonic amplitude is zero, so the solution contains only terms at the forcing frequency 2ω and its harmonics. The stability of the forced solution and the conditions under which the subharmonic will build up from rest are determined by investigating the stability of the equilibrium point given by $a = 0$. If the solution $a = 0$

is stable, the forced solution is stable and the subharmonic will not build up from rest. Conversely, if the solution a = 0 is unstable, the subharmonic can build up.

In order to investigate the stability of this solution, let

$$a = 0 + \alpha$$

and (100)

$$2\,\phi = 2\,\phi_o + \eta = \cos^{-1}\frac{3\omega^2 h}{F} + \eta \quad .$$

When these expressions are substituted into equations (89) and (90), the variational equations are given by

$$\frac{d\alpha}{dt} = \frac{\alpha\epsilon^2}{2\omega}\left(-\,k\omega + \frac{F}{3\omega^2}\sin 2\,\phi_o\right) \tag{101}$$

$$= \frac{\alpha\epsilon^2}{2\,\omega}\left(-\,k\omega + \frac{F}{3\omega^2}\sqrt{1 - \left(\frac{3\omega^2 h}{F}\right)^2}\right)$$

and

$$\frac{d\eta}{dt} = -\,\frac{\epsilon^2 F}{6\omega^3}\,\eta\,(\sin 2\,\phi_o)\ . \tag{102}$$

Equation (101) shows that $\frac{d\alpha}{dt}$ is negative and hence the condition a = 0 is stable if

$$k > \frac{F}{3\omega^3}\sqrt{1 - \left(\frac{3\omega^2 h}{F}\right)^2}\ ,$$

or unstable if

$$k < \frac{F}{3\omega^3}\sqrt{1 - \left(\frac{3\omega^2 h}{F}\right)^2}\quad .$$

In other words, the subharmonic threshold is unstable if

$$\frac{F^2}{9\,\omega^4} > k^2\,\omega^2 + h^2 \quad .$$

In terms of the original constants, the subharmonic will build up from rest if

$$I_o^{\,2}|Z|^2\ \frac{D_1\Omega_o^{\,4}}{9\omega^4L^2} > \omega^2\,\frac{(R_L + R_c)^2}{L^2} + (\Omega_o^{\,2} - \omega^2)^2. \tag{103}$$

A comparison of equations (94) and (103) shows that the subharmonic can continue to exist for a frequency $(\omega < \Omega_o)$ at which it will not build up from rest.

The capacitor charge of Figure 12(a) is given to a second approximation by

$$q = a\,\sin(\omega t + \phi) - \frac{D_1\Omega_o^{\,2}}{\omega^2}\left[\frac{a^2}{2} + \frac{a^2}{6}\cos(2\omega t + 2\phi)\right] \tag{104}$$

$$- \frac{I_o|Z|}{3L\omega^2}\cos 2\,\omega t \ ,$$

where a and ϕ are given by equations (94) and (95) respectively.

Summary of properties of analytical solutions--The results of this chapter are summarized below. A comparison is given between the solutions derived by the two methods.

Equation (104) shows there is always a periodic solution

$$q = -\frac{I_o|Z|}{3L\,\omega^2}\cos 2\,\theta = -\frac{I_o}{3\,\omega^2 L}\sqrt{R_L^{\,2} + (\omega L)^2}\,\cos 2\,\omega\,t \quad .$$

This solution is stable unless

$$\frac{\Omega_o^4 D_1^2 I_o^2 |Z|}{9\omega^4 L^2} > \frac{(R_L + R_c)^2}{L^2} \omega^2 + (\Omega_o^2 - \omega^2)^2 .$$

According to (94), (95) and (99), there are subharmonics if

$$\frac{I_o |Z| D_1 \Omega_o^2}{3\omega^2} > (R_L + R_c)\,\omega$$

and

$$\omega^2 - \Omega_o^2 < \sqrt{\left(\frac{I_o |Z| D_1 \Omega_o^2}{3\omega^2 L}\right)^2 - \frac{(R_L + R_c)^2}{L^2}\,\omega^2} .$$

There are two subharmonics with a phase difference π, if

$$|-\omega^2 + \Omega_o^2| > \sqrt{\left(\frac{I_o |Z| D_1 \Omega_o^2}{3\omega^2 L}\right)^2 - \frac{(R_L + R_c)^2}{L^2}\,\omega^2} ,$$

and there are four subharmonics, one stable pair and one unstable pair, if

$$-\omega^2 + \Omega_o^2 > \sqrt{\left(\frac{I_o |Z| D_1 \Omega_o^2}{3\omega^2 L}\right)^2 - \frac{(R_L + R_c)^2}{L^2}\,\omega^2} .$$

In each case the solution as given by equation (104) is

$$q = a \sin(\omega t + \phi) - \frac{I_o |Z|}{3\omega^2 L} \cos 2\omega t - D_1 \frac{\Omega_o^2}{\omega^2}\left[\frac{a^2}{2} + \frac{a^2}{6}(2\omega t + 2\phi)\right] ,$$

where from equations (94) and (95)

$$a^2 = \frac{6\omega^2}{5D_1^2\Omega_o^4}\left[-(\omega^2 - \Omega_o^2) \pm \sqrt{\left(\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}\right)^2 - \left(\frac{R_L + R_c}{L}\right)^2 \omega^2}\,\right]$$

and

$$2\,\phi = \sin^{-1}\frac{3\omega^3(R_L + R_c)}{D_1\Omega_o^2 L I_o|Z|} \quad .$$

The positive sign of the equation for a^2 yields the stable pair of subharmonics and the negative sign gives the unstable pair.

If the amplitude is varied with a fixed exciting frequency, ω , the solutions vary as follows:

(a) As I_o increases from 0 to

$$\frac{3(R_L + R_c)}{|Z|D_1\Omega_o^2}\ \omega^3 \quad ,$$

there is a stable periodic solution of period π, and there are no subharmonics.

(b) If $\Omega_o > \omega$, two pairs of subharmonics appear when I_o reaches

$$\frac{3\omega^2}{|Z|D_1\Omega_o^2}\ (R_L + R_c) \quad ,$$

one pair stable and one pair unstable with amplitudes given by (94). The periodic solution remains stable.

When

$$\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}$$

reaches

$$\left[\left(\frac{R_L + R_c}{L}\right)^2 \omega^2 + (\omega^2 - \Omega_o^2)^2\right]^{1|2}$$

the unstable pair of subharmonics disappears, the stable pair remains, and the periodic solution becomes unstable.

(c) If $\Omega_o < \omega$ no subharmonics appear until

$$\frac{I_o|Z|D_1\ \Omega_o^2}{3\ \omega^2 L}$$

reaches

$$\left[\left(\frac{R_L + R_c}{L}\right)^2 \quad \omega^2 + (\omega^2 - \Omega_o^2)^2\right]^{1|2};$$

a stable pair of subharmonics then appears and the periodic solution becomes unstable.

If the exciting frequency, ω , is varied with fixed input current, I_o, the so-called tuning curves for the solutions are obtained.

(a) If

$$R_L + R_c > \frac{I_o|Z|D_1\ \Omega_o^2}{3\ \omega^3} \quad ,$$

there is a stable periodic solution and there are no subharmonics.

(b) If

$$R_L + R_c < \frac{I_o|Z|D_1\Omega_o^2}{3\,\omega^3} \quad ,$$

there are three ranges of ω which are of interest. If

$$\Omega_o^2 - \omega^2 < -\sqrt{\left(\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}\right)^2 - \frac{(R_c + R_L)^2}{L^2}\,\omega^2} \quad ,$$

there is a stable periodic solution and there are no subharmonics. If

$$|\Omega_o^2 - \omega^2| < \sqrt{\left(\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}\right)^2 - \left(\frac{R_c + R_L}{L}\right)^2 \omega^2} \quad .$$

the periodic solution is unstable and there is a stable pair of subharmonics. If

$$\Omega_o^2 - \omega^2 > \sqrt{\left(\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}\right)^2 - \left(\frac{R_c + R_L}{L}\right)^2 \omega^2} \quad ,$$

there is a stable periodic solution, a pair of stable subharmonics, and a pair of unstable subharmonics.

Curves of amplitude plotted against I_o(with ω constant) or ω(with I_o constant) are sketched in Figures 13 and 14. Unstable solutions are indicated by dotted lines. Figure 13 reveals the possibility of a "hysteresis effect", when $\Omega_o > \omega$, if I_o is increased and then decreased again. As I_o increases, the stable periodic solution exists until

$$\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}$$

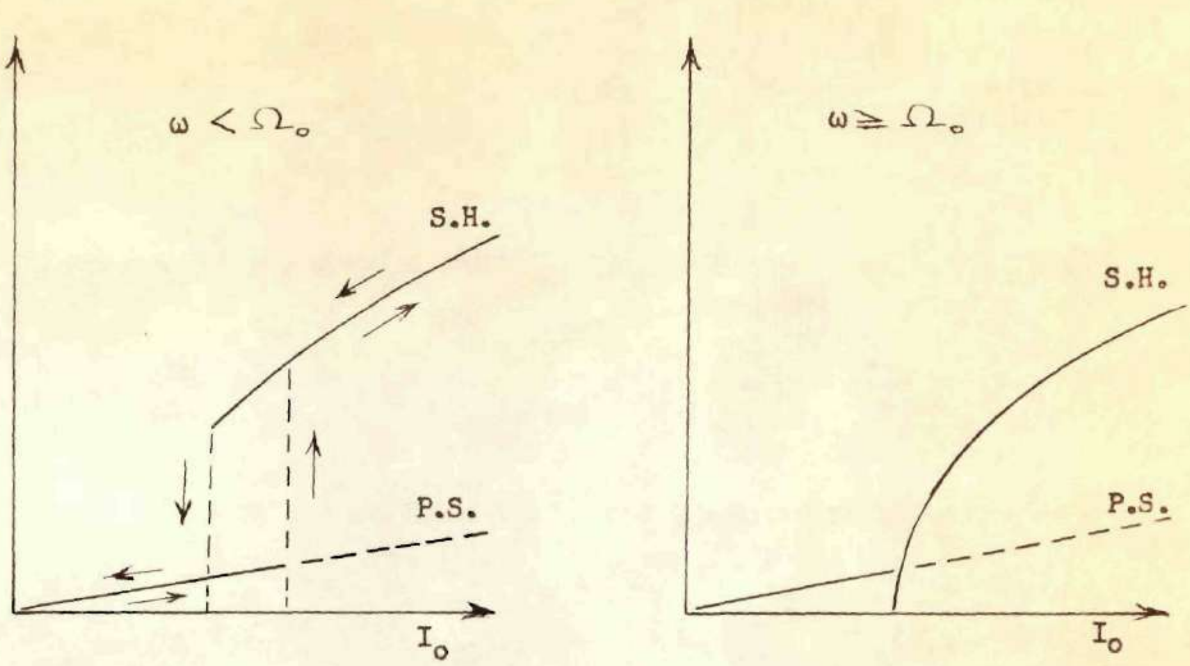


FIG. 13. HARMONIC AND SUBHARMONIC AMPLITUDES AS FUNCTIONS OF EXCITING CURRENT

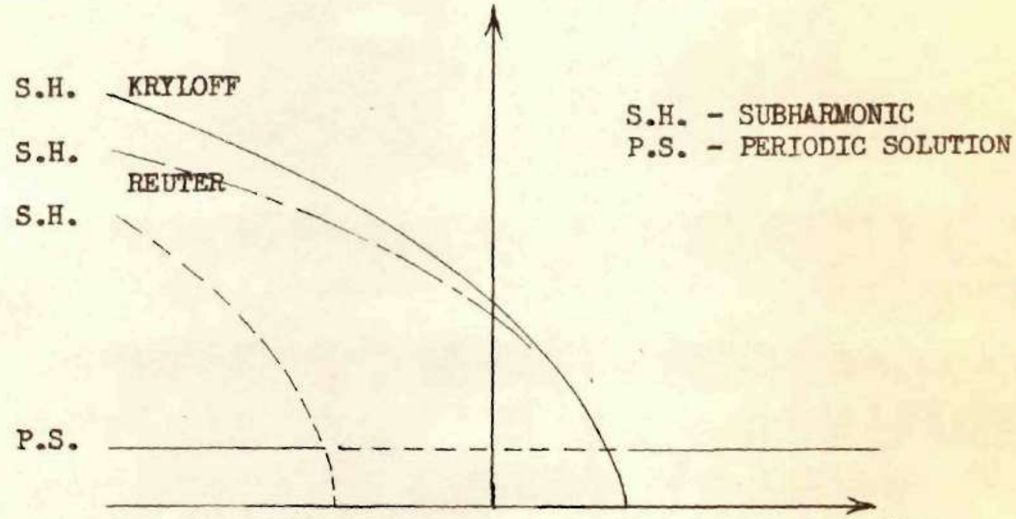


FIG. 14. VARIATION OF SUBHARMONIC AMPLITUDE WITH TUNING

reaches

$$\sqrt{\left(\frac{R_c + R_L}{L}\right)^2 \omega^2 + (\Omega_o^{\,2} - \omega^2)^2}\ ,$$

where there is a jump to the subharmonic solution. If now I_o is decreased, the subharmonic solution continues until

$$\frac{I_o |Z| D_1 \Omega_o^{\,2}}{3\omega^2}$$

reaches $(R_c + R_L)\omega$, when a jump back to the periodic solution takes place.

The second approximation, equation (94), and the perturbation series result, equation (82), are sketched in Figure 14.

A comparison of equations (82) and (94) for the subharmonic amplitude shows that the two equations yield the same result, if $\omega = \Omega_o$. Equations (83) and (95) show that $2\phi = \beta + \frac{\pi}{2}$ due to the different forms of the phases of the zeroth approximations. For $\omega - \Omega_o \neq 0$, equations (82) and (94) differ by multiplying factors

$$\frac{\Omega_o^{\,2}}{\omega^2}$$

on I_o and the total a^2 term. Since $\Omega_o^{\,2} - \omega^2$ is of order ϵ^2, the difference of the two results is of small order. Regarding this discrepancy, it can be argued that there should be a lower frequency limit, that is not predicted by either analysis, to the region in which the second order subharmonic exists. Since this lower limit, ω_x, is such that $\Omega_o^{\,2} - \omega_x^{\,2}$ is of order ϵ, the analysis does not hold at the lower frequency limit and neither result describes accurately the lower frequency limit of the subharmonic solution. This point will be discussed further later in this section when computed responses are compared with experimental results.

The stability conditions and properties of the solutions obtained by the second approximation are the same as those obtained by Reuter[38] with perturbation series, except for the factor $\frac{\Omega_o^2}{\omega^2}$ discussed above. It is informative to compare the methods of solution, however. In the approximation method, the amplitude and phase of the subharmonic solution were obtained directly in terms of the small parameter ϵ. However, the cosine and sine components of the solution were obtained by the perturbation method. It appears that generally less mathematical manipulation is required to obtain a solution by the approximation method and certainly the application of stability criteria is more straightforward. The straightforward application of perturbation series often requires careful interpretation of the coefficients ω_1 and ω_2, while this difficulty does not arise in the approximation method. It is believed that the successive approximation method is more readily useful to most engineers and allows greater ease of computation, particularly if higher order approximations are necessary.

Effects of Third-Order Curvature on the Subharmonic Solution of Second Order

The data of Chapter III show that unbiased dielectrics possess odd order curvature in their charge-voltage characteristic. Thus it is necessary to bias the ferroelectric element to achieve the even order curvature required for the existence of even order subharmonics. Since some odd order curvature will exist with a biased element, it is necessary to examine the effects of higher order curvature on the second order subharmonic. When both second and third order curvatures exist in the charge-voltage characteristic both D_1 and D_2 are nonzero in the polynomial which approximates the element characteristic, and the capacitor voltage, v, is approximately,

$$v = \frac{1}{C_o} [1 + D_1 q + D_2 q^2] \, q \, .$$

Now D_1 is the coefficient of the desired curvature, while D_2 is undesired for the second order subharmonic. Let $D_2 \Omega_o^{\ 2}$ be of order ϵ^2, where $D_1 \Omega_o^{\ 2}$ is of order ϵ. If the loss and detuning are of order ϵ^2, then from equations (28) and (85) we have

$$\epsilon f_1(q, \omega q') = D_1 \, \Omega_o^{\ 2} q^2$$
$$\epsilon^2 f_2(q, \omega \, q) = \omega\left(\frac{R_c + R_L}{L}\right) q + (\Omega_o^{\ 2} - \omega^2) q + \Omega_o^{\ 2} D_2 q^3 . \qquad (105)$$

Since the added term is of order ϵ^2, the first approximation to the solution is still given by equation (86) and $\sigma_1 = A_1 = 0$. The second approximation is obtained from an equation analogous to equation (87) but including the $D_2 q^3$ term. If

$$\epsilon F = \frac{I_o |Z|}{L} \quad ,$$

this equation is

$$\frac{\partial^2 Z_2}{\partial \theta^2} + Z_2 = 2 \frac{\sigma_2}{\omega} a \sin(\theta + \phi) - 2 \frac{A_2}{\omega} \cos(\theta + \phi) \qquad (106)$$
$$+ \frac{1}{\epsilon^2} \left\{ - \frac{(R_c + R_L) a\omega}{L} \cos(\theta + \phi) - \left(\frac{\Omega_o^{\ 2}}{\omega^2} - 1 \right) a \sin(\theta + \phi) \right.$$
$$+ 2a \frac{D_1^{\ 2} \Omega_o^{\ 4}}{\omega^4} \sin(\theta + \phi) \left[\frac{I_o |Z|}{3 D_1 \Omega_o^{\ 2}} \cos 2\theta + \frac{a^2}{2} + \frac{a^2}{6} \cos 2(\theta + \phi) \right]$$

$$+ \ \Omega_o^{\,2} \frac{D_2 a^3}{4\omega^2} \left[- \sin 3(\theta + \phi) + 3\sin(\theta + \phi) \right].$$

The bracketed terms of equation (106) are of the order ϵ^2 so that $\frac{1}{\epsilon^2}$ times these terms is of order unity. The sum of the sin $(\theta + \phi)$ and $\cos(\theta + \phi)$ terms of equation (106) must be zero in order that Z_2 be periodic. σ_2 and A_2 are then given by

$$\sigma_2 \epsilon^2 = -\frac{1}{2\omega}\left[\omega^2 - \Omega_o^{\,2} + \frac{5a^2}{6\omega^2} D_1^{\,2}\, \Omega_o^{\,4} + \frac{3\Omega_o^{\,2} D_2 a^2}{4} - \frac{I_o |Z| D_1}{3\omega^2 L}\, \Omega_o^{\,2} \cos 2\phi\right],$$

and

$$A_2\, \epsilon^2 = \frac{a}{2\omega}\left[-\frac{(R_L + R_c)}{L}\,\omega + \frac{I_o |Z| D_1 \Omega_o^{\,2}}{3\,\omega^2 L} \sin 2\,\phi\right].$$

Thus by equations (39), since $A_1 = \sigma_1 = 0$,

$$\frac{da}{dt} = A_2 \epsilon^2 = \frac{a}{2\omega}\left[-\frac{(R_L + R_c)}{L}\,\omega + \frac{I_o |Z| D_1 \Omega_o^{\,2}}{3\,\omega^2 L} \sin 2\,\phi\right] \tag{107}$$

$$\frac{d\phi}{dt} = -\frac{1}{2\omega}\left[\omega^2 - \Omega_o^{\,2} + \frac{a^2}{2}\left(\frac{5 D_1^{\,2} \Omega_o^{\,4}}{3\,\omega^2} + \frac{3 D_2 \Omega_o^{\,2}}{2}\right) - \frac{I_o |Z| D_1 \Omega_o^{\,2}}{3\,\omega^2 L} \cos 2\,\phi\right]. \tag{108}$$

The equilibrium conditions $\dot{a} = 0$ and $\dot{\phi} = 0$ are satisfied, if

$$2\phi = \sin^{-1} \frac{3\,\omega^3 (R_L + R_c)}{I_o |Z| D_1 \Omega_o^{\,2}}$$

and

$$a^2 = \frac{6\omega^2}{5D_1\Omega_o^4}\left(1+\frac{9D_2\,\omega^2}{10D_1^2\Omega_o^2}\right)^{-1} \times \tag{109}$$

$$\left[\Omega_o^2 - \omega^2 \pm \sqrt{\left(\frac{I_o|Z|D_1\Omega_o^2}{3\omega^2 L}\right)^2 - \left(\frac{R_L + R_c}{L}\right)^2\omega^2}\,\right].$$

Equation (104) differs from (94) only by a multiplying factor which tends to reduce the amplitude. The stability and existence conditions are changed only in details and need not be considered further.

If the D_2 and D_1 terms are of the same order, the subharmonic solution can be investigated by the second approximation method, as used above. However, if the D_2 term is not of order ϵ^2 then $A_1 \neq 0$ and the final expression which defines the amplitude is a high order algebraic equation so that an explicit solution for a is obtainable only numerically.

It is also possible to derive a second approximation solution when the capacitor loss resistance varies with charge, as $R_c = R_o(1 + aq)$, where R_o and a R_o are of order ϵ^2. Loss variations are a second or third order effect in the experimental circuits of the next section and hence such a solution is not justified in view of the experimental errors in establishing tuning conditions and the coefficients D_1, D_2 and C_o.

A comparison of equations (68) and (69) shows that the analysis of this section will apply to the voltage-fed circuit of Figure 12(b) if the substitution

$$E = I_o|Z| \tag{110}$$

is made in the equations of this section.

Experimental and Calculated Results on Second Order Subharmonics

It is possible to generate the second order subharmonic in either a voltage-fed circuit or a current-fed circuit as shown in Figure 12. If $\omega = \Omega_o$, from equations (94)

$$I_o \geq \frac{3(R_c + R_L)}{|Z| D_1} = \frac{3\,\omega}{2D_1 Q}$$

and

$$E \geq \frac{3(R_c + R_L)\omega}{D_1} = \frac{3\,\omega^2 L}{D_1\;Q}$$

are the conditions for the existence of the second subharmonic in current and voltage-fed circuits, respectively. For a circuit having an inductance of 0.6 henry, a quality factor (Q) of 20, a resonant frequency of 4 kilocycles and a dielectric with $D_1 = 0.5 \times 10^6$ (a typical value for the ET61 dielectric) the required current and voltage are approximately 3.78 milliamperes and 114 volts. The voltage must be delivered from a low impedance source. At audio or higher frequencies a high impedance current source is more easily realized than a low impedance voltage source. Thus the current-fed circuit is generally of more interest.

An experimental current-fed circuit was set up using a pentode amplifier tube as the current source. This circuit is shown in Figure 15. The nonlinear current and tube plate current were measured by reading the voltages developed across the low resistors R and R_p, respectively. A harmonic wave analyzer was used to read the currents and voltages, so that the amplitudes of the separate frequency components could be measured.

The temperature dependence of the nonlinear dielectrics led to experimental difficulties. Since dielectrics are moderately lossy, a

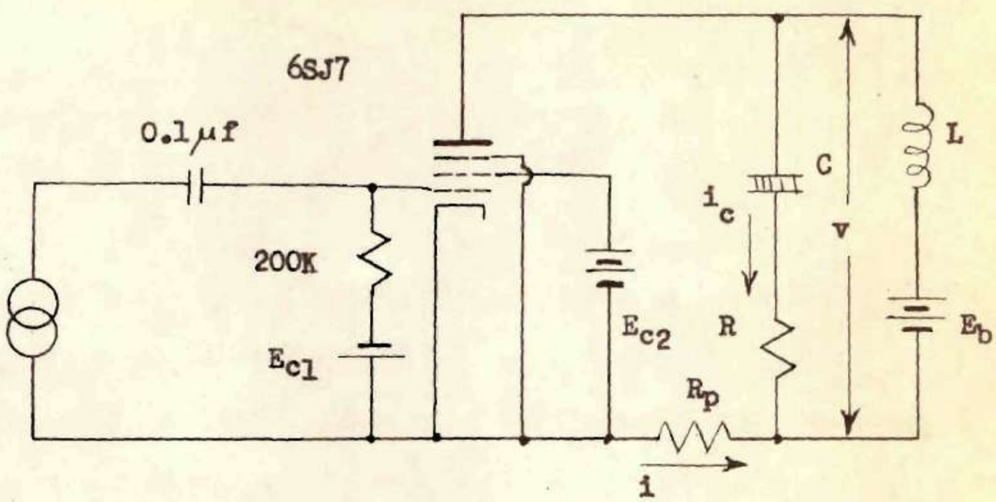


FIG. 15. CIRCUIT FOR EXPERIMENTAL STUDY OF SUBHARMONICS

significant power loss occurs in the capacitor if a circulating current exists in the resonant circuit. This power loss causes heating and a change of the capacitor characteristics. It was shown in Chapter II that the capacitors even become linear at very high temperatures. The small plate area of capacitors with these high dielectric constant materials allows very little cooling by convection and hence the nonlinear capacitors must be cooled by conduction. In order to minimize the temperature changes of the dielectrics the nonlinear capacitors were prepared by silver plating each sample. The plated units were then either soldered to a metallic heat conductor or immersed in an oil bath. A low temperature solder was used in all soldering operations to minimize thermal shock on the dielectrics. The importance of proper cooling is emphasized by the fact that the soldered joint melted on one early unit in which small copper leads only were used to provide cooling. The thinness of the samples leads to a tendency for surface arcs to occur around its edges. These arcs can be avoided by coating capacitors with a polystyrene dope. This is unnecessary if the unit is to be immersed in an insulating oil bath.

Another experimental difficulty arises in tuning the tank circuit to achieve a zero detuning condition for the subharmonic. The hysteresis loops of Figure 8 and the harmonic analyzer data of Table 3 show that the apparent slope at the bias point of the charge-versus-voltage characteristic varies somewhat with applied alternating voltage. This results from the finite width of the hysteresis loop and shows that the incremental capacity is not equal to the slope of the charge-versus-voltage characteristic for large signals. Two methods of tuning were adopted and these gave consistent results and resulted in approximately the same frequency. These techniques consist of either determining the frequency at which a minimum signal is required to excite the subharmonic or tuning the circuit so as to obtain a maximum tank voltage for a moderate amplitude of excitation. That is, the circuit could be tuned by maximizing the response to a driving signal of 50 volts across the capacitor. The tuning method used will be given with each set of experimental data. The maximum response method alone was used in measurements on multiply resonant circuits. Due to the above mentioned tuning difficulty the experimental tuning curves are in some cases displaced a constant amount in frequency from the true conditions.

In studying the subharmonic response as a function of frequency the difference between the actual and the reference frequencies was measured by feeding to a nonlinear mixer the outputs of two audio signal generators, one generator at the reference frequency and the other at the excitation frequency. The mixer output was applied to an oscilloscope and the difference frequency determined by Lissajous patterns.

Capacitors of the ET61, ET46 and K3300 dielectrics were prepared and placed in an oil bath. Subharmonic responses for each of these capacitors were measured. In each case the reference frequency was taken as that frequency which required a minimum current input to excite the second subharmonic. Data on the characteristics of these three capacitors taken by the sine-wave-of-charge harmonic analysis of capacitor voltage are given in Table 3 and hysteresis loops of charge versus voltage for the ET46 and ET61 dielectrics are given in Figures 6 and 9 of Chapter III. The data on these capacitors show that the nonlinear characteristic can be approximated roughly by the polynomials

$$\frac{1}{C} = \frac{10^9}{2.7}\,(1 + 0.48 \times 10^6\, q + 0.19 \times 10^{12} q^2) \text{ for ET61 \#3}\,, \tag{111}$$

$$\frac{1}{C} = \frac{10^9}{2.8}\,(1 + 0.42 \times 10^6\, q + 0.16 \times 10^{12} q^2) \text{ for ET46}\,, \tag{112}$$

$$\frac{1}{C} = \frac{10^9}{4.4}\,(1 + 0.27 \times 10^6\, q + 0.16 \times 10^{12} q^2) \text{ for K3300}\,. \tag{113}$$

With a tank inductance of 0.67 henry and a reference frequency of 4.12 kc. the tank Q was measured on a Freed Q Indicator and found to be approximately 13 with 13 volts across the capacitor from the Q meter. A moderate voltage from the Q meter was used deliberately in order to include an approximation to the hysteresis loss: for example, in this case the Q decreased from 15 with 1.5 volts to 13 with 13 volts. Table 5 gives the measured data on the voltages and currents for the ET61 sample No. 3 capacitor for the case when the input frequency is twice the reference frequency, which is the condition of zero detuning. Plots of periodic and subharmonic components of current against the fundamental component of plate current, I_o, at a frequency 2 f_r are given in Figure 16. Tuning data are given in Figure 17 and Table 6, which show the variation of current and voltage as the input frequency is varied. In Figure 17 the difference frequency is one-half the input frequency minus the reference frequency. The pairs of vertical lines on the frequency scale indicate the frequency limits for which the theoretical assumptions are valid. Some heating of the dielectric was observed in these tests as evidenced by the fact that the curves would repeat themselves for increasing and decreasing currents only if the capacitor was allowed to cool. In each case the threshold condition of the subharmonic was observed to be somewhat unstable in that the subharmonic amplitude varied considerably at the threshold. This instability disappeared as the driving current

TABLE 5

SUBHARMONIC AMPLITUDE RESPONSE WITHOUT DETUNING

ET61 #3, $f_r = 4.15$ kc, $L = 0.67$ h, $\Delta F = 0$

$Q \doteq 13$, $E_b = E_{c_2} = 250$, $E_{c_1} = -5$, 76° F .

e_s volts at 2 f_r	I_o ma. at 2 f_r	I_c--Capacitor Current--ma. at f_r	at $2f_r$	at $3f_r$	Capacitor Voltage v at f_r	at $2f_r$
1.0	2.6	0	2.3	0	0	23.8
1.5	3.9	0	3.45	0	0	35.7
1.6		Threshold				
1.62	4.16	0.4	3.7	0.01	6.7	38.5
1.75	4.40	1.6	3.9	0.04	30.1	42.1
2.00	5.10	2.9	4.45	0.09	58.0	46.8
2.50	5.80	4.6	5.4	0.18	95.2	58.8
3.00	6.70	6.0	6.4	0.25	119.0	73.0
3.50	8.50	7.3	7.6	0.37	150	95.2

All currents and voltages are r.m.s. values.

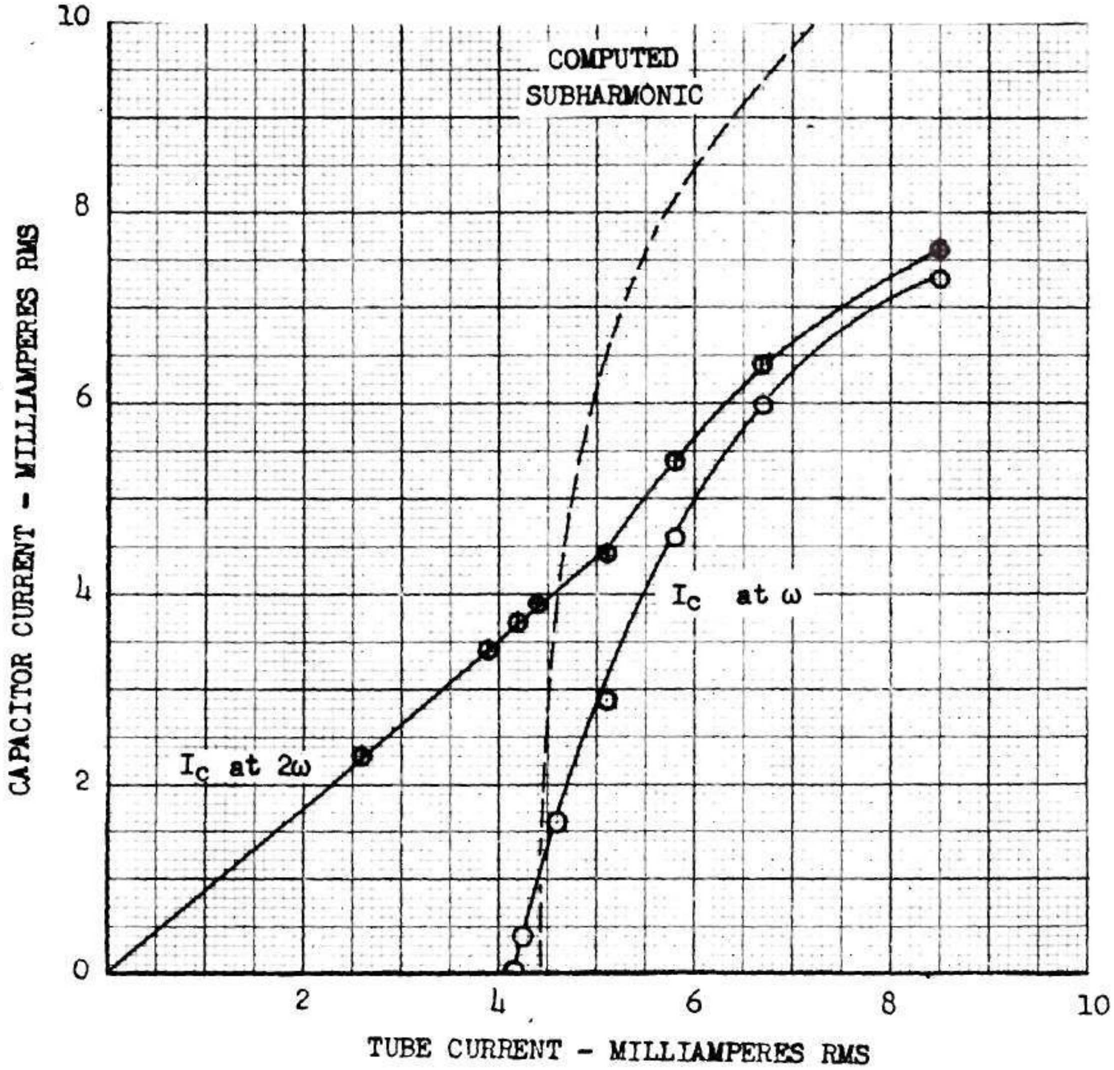


FIG. 16. CAPACITOR CURRENT VERSUS TUBE CURRENT FOR SECOND ORDER SUBHARMONIC

TABLE 6

SECOND SUBHARMONIC TUNING DATA

ET61 #3, $f_r = 4.15$ kc, $L = 0.27$ h, $Q = 13$,

$E_b = E_{c_2} = 250$, $E_{c_1} = -5$, $e_s = 2.5$ volts r.m.s., 76° F

$I_o = 6.2$ ma. r.m.s.

$\Delta F = f - f_r$ c.p.s.	I_c ma. r.m.s. at f	at 2f	v r.m.s. volts at f	at 2f
108	0	5.8	0	57.1
99	1.4	5.8	28.6	58.6
85	2.5	5.6	53.1	61.9
58	4.2	6.1	79.4	62.5
35	5.1	6.1	102	66.6
0	5.4	6.2	113	69.0
-21	6.4		127	
-49	6.7		135	
-98	7.7		151	
-148	8.0		155	
-193	7.6		147	
-224	6.8		127	
-239	0	6.3	0	66.6

No measureable hysteresis.

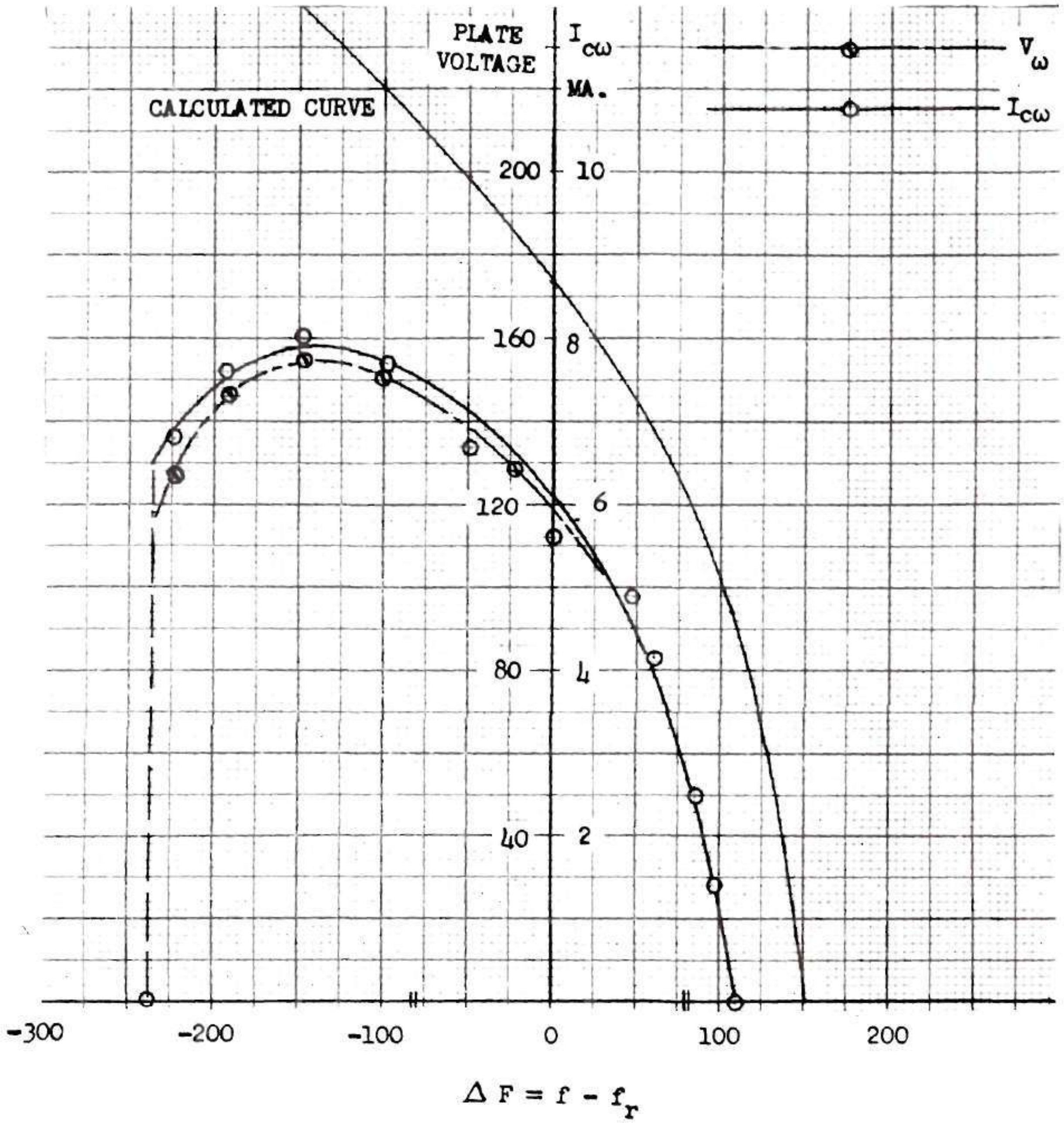


FIG. 17. SECOND SUBHARMONIC TUNING CURVES

increased above the threshold condition.

With the values used in these tests on the ET61 Sample No. 3 capacitor, the computed subharmonic solution, equation (109), gives the r.m.s. milliamperes of subharmonic capacitor current in terms of the r.m.s. milliamperes of tube current for zero detuning as

$$I_{c_{r.m.s.}} = 6.26 \left[0.2\, I^2_{o_{r.m.s.}} - 4 \right]^{1/4} \quad \text{ma} ,$$

with L = 0.67 henry,

$$Q = \frac{\omega_r L}{R_c + R_L} = 13,$$

$f_r = 4.15$ kc, $D_1 = 0.48 \times 10^6$, and $D_2 = 0.19 \times 10^6$. This theoretical curve is plotted in Figure 16 along with $f = f_r$ experimental response curve.

Each of the curves of the periodic solution takes on additional curvature when the subharmonic amplitude becomes appreciable. This is to be expected since the harmonic wave analyzer measures the total signal at a particular frequency. Therefore, the curves at 2f will include the second harmonic of the subharmonic as well as the periodic solution itself. Thus the capacitor current or voltage at the driving frequency 2f is a linear function of tube current up to the point at which the subharmonic commences and thereafter is the sum of the linear forced response and a second harmonic of the subharmonic. By equation (104) the capacitor current at the forcing frequency is

$$\dot{q} = \frac{D_1 \Omega_o^{\,2}}{3\,\omega}\; a^2 \sin(2\omega t + 2\,\phi) + \frac{2 I_o |Z|}{3\,L\,\omega} \sin 2\omega t \quad ,$$

where a and ϕ are the amplitude and phase of the subharmonic component of charge given by equation (109). For a low loss tank circuit the subharmonic charge is practically in phase with the driving current so ϕ is approximately zero. If $\omega = \Omega_o$, the approximate amplitude of the capacitor current at the forcing frequency is given by

$$I_{2\omega} = D_1 \frac{\omega a^2}{3} + \frac{4I_o}{3}$$
$$= \frac{4I_o}{3} + \frac{D_1}{2\omega}\left(1 + \frac{9D_2}{10D_1^2}\right)^{-1}\left[\frac{6}{5D_1^2}\sqrt{\left(\frac{2I_oD_1\omega}{3}\right)^2 - \left(\frac{R_c + R_L}{L}\right)^2 \omega^2}\right]. \qquad (114)$$

Equation (114) correctly predicts the type of curvature shown in the experimental curve of Figure 16. However, the measured capacitor current at 2f is less than that predicted by (114), even when a = 0. Thus it appears that calibration errors may exist in the measured capacitor or tube currents.

The capacitor voltage is related to the charge by the polynomial approximation to its characteristic, which in this case is

$$v = \frac{q}{C_o}\left(1 + D_1 q + D_2 q^2\right).$$

The magnitude of the subharmonic and forced frequency components of the capacitor voltage are then given approximately by

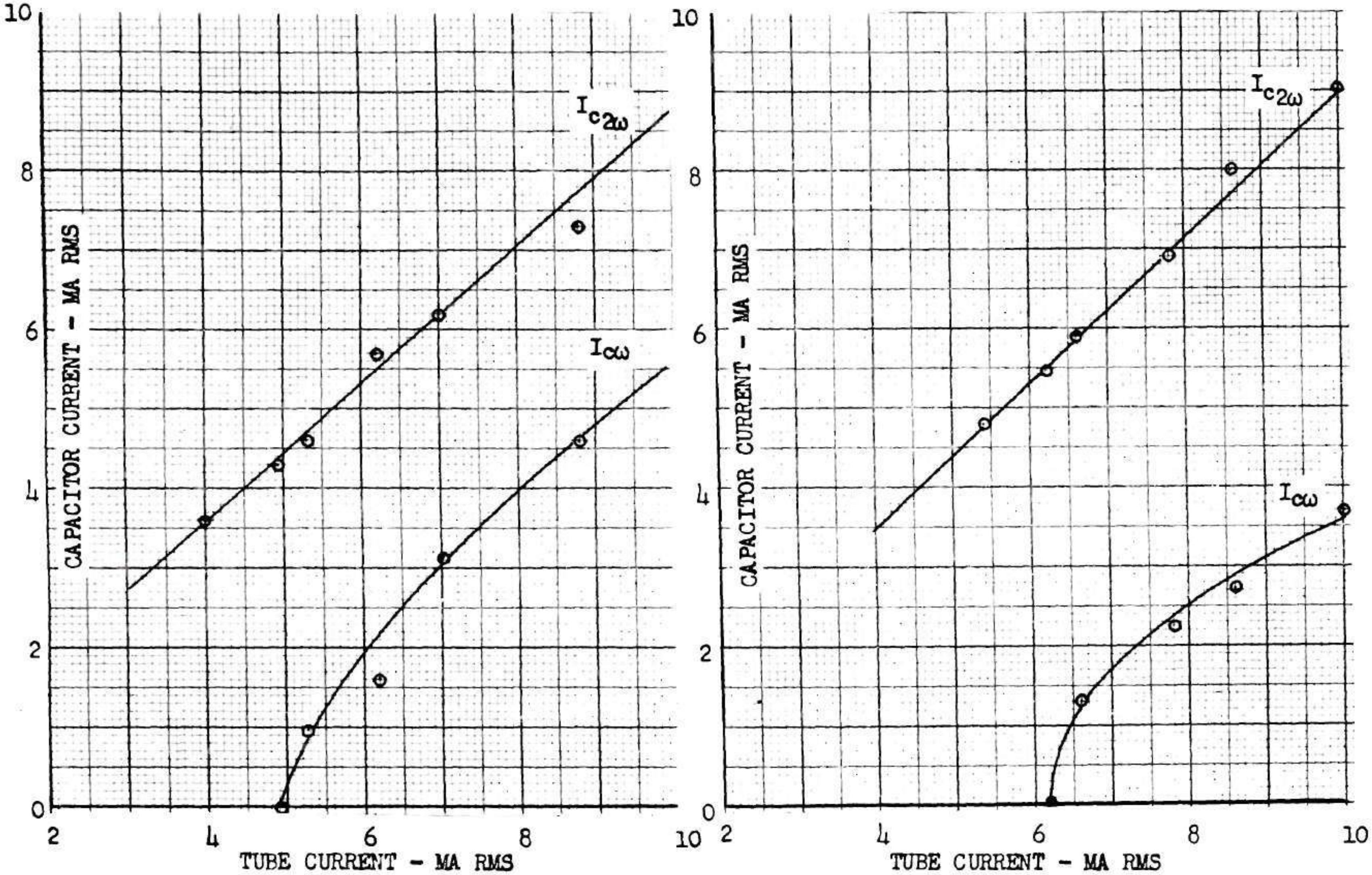
$$v_\omega = \frac{a}{C_o}\sin(\omega t + \phi)$$

$$v_{2\omega} = -\frac{I_o|Z|}{3LC_o\omega^2}\cos 2\omega t - \frac{D_1\Omega_o^2 a^2}{6\omega^2 C_o}\cos 2(\omega t + \phi)$$

where a is given by equation (109) and only first order terms are retained.

The ET46 and K3300 dielectric capacitors were placed in the oil bath and similar data on the current and voltage responses taken. Because of the fragility of the 0.01 inch thick K3300 dielectric, this capacitor was formed by clamping the dielectric in a pressure-type crystal holder rather than soldering leads directly to the silver plating of the dielectric as was done with the ET61 and ET46 dielectrics. Amplitude response curves, for $f = f_r$, and tuning curves for these units are given in Figures 18 and 19. The ET46 unit was operated with E_b = 250 volts, f_r = 4.15 kc, and L = 0.67 henries. Under these conditions the tank Q was 10 with 10 volts across the capacitor. The K3300 unit was operated with E_b = 250 volts, L = 0.67 henries, and f_r = 3.25 kc. The tank Q was 15 with 15 volts across the capacitor. The tests on both the K3300 and ET46 indicated that some local heating occurred in the dielectrics. In the ET46 detuning run, the data failed to repeat at zero detuning even after some minutes were allowed for the capacitor to cool. It appeared that this particular sample was altered in its characteristics and the resonant frequency shifted to 3.95 kc. This permanent change of properties was not observed in any other samples of the dielectrics tested.

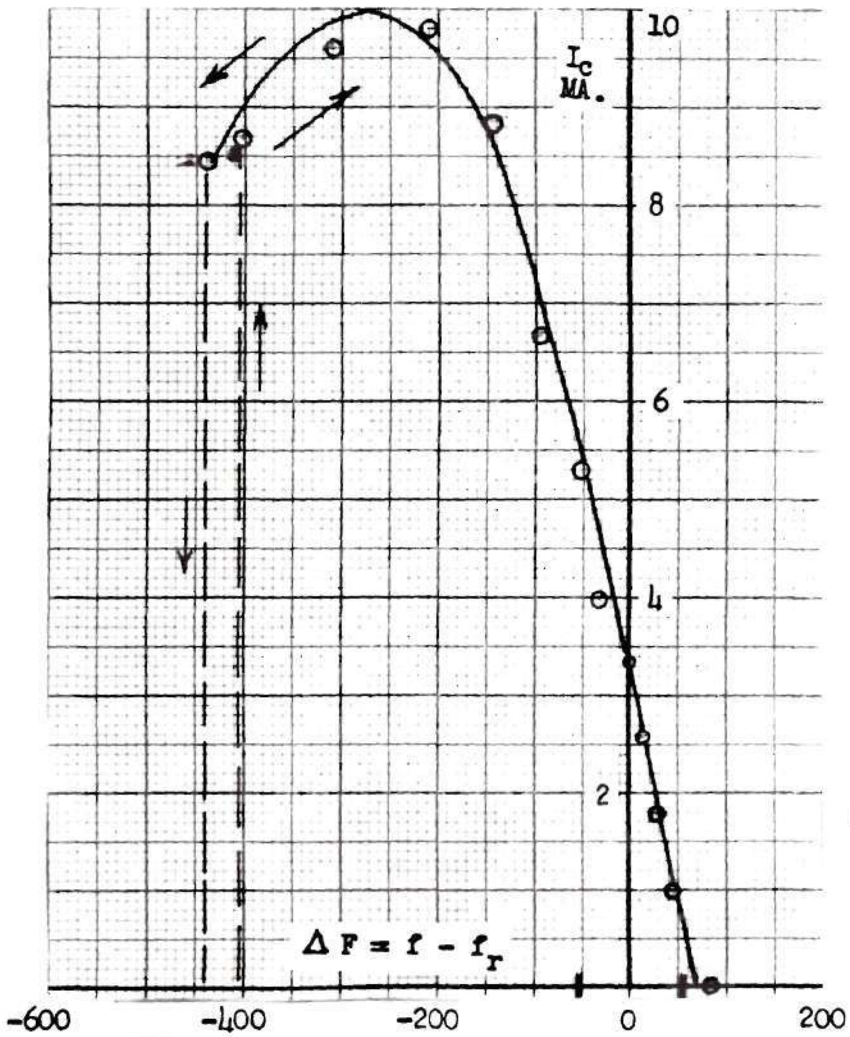
If it is assumed that the experimental reference frequency is equal to the resonant frequency of the tank circuit, the theoretical subharmonic capacitor current can be plotted as a function of the frequency difference $\Delta F = f - f_r$, which is assumed equal to $f - f_o$. From equation (109) the amplitude of the subharmonic capacitor current is



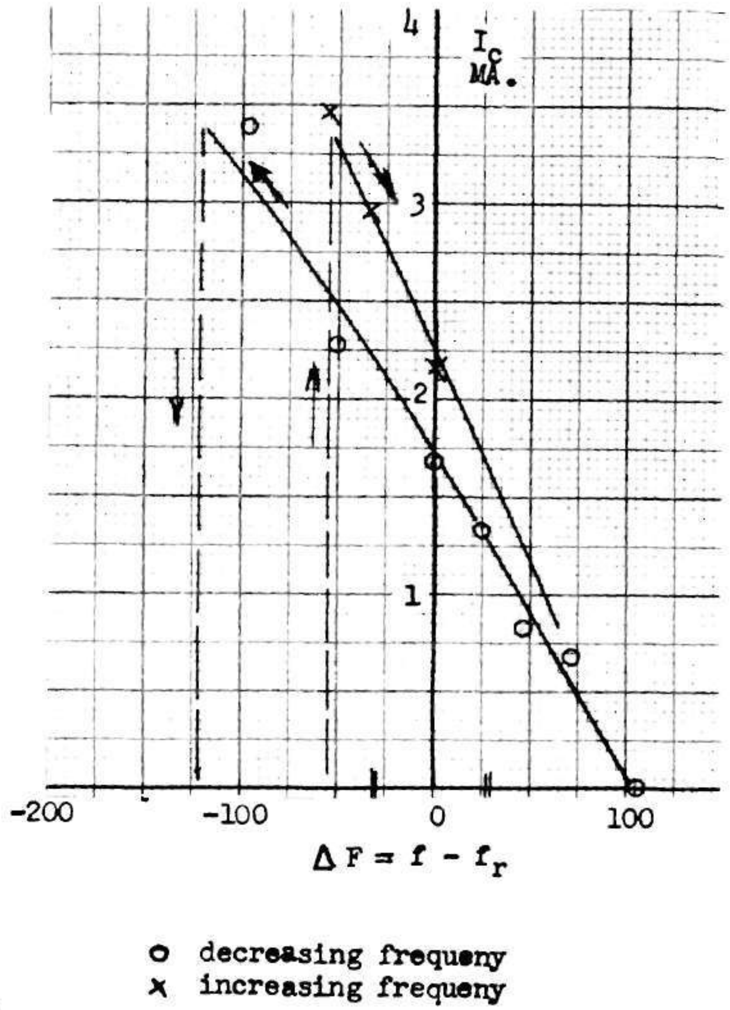
A. ET 46 DIELECTRIC

B. K3300 DIELECTRIC

FIG. 18. SECOND SUBHARMONIC RESPONSE CURVES WITH ZERO DETUNING



A. ET46, f_r = 4.15 kc, Q = 10
e_s = 3.0 volts



B. K3300, f_r = 3.25 kc, Q = 15
e_s = 3.0 volts

FIG. 19. SECOND SUBHARMONIC TUNING CURVES

$$I_c = \left\{ \frac{6\omega^4}{5D_1{}^2\Omega_o{}^4} \left(1 + \frac{9D_2\omega^2}{10D_1{}^2\Omega_o{}^2} \right)^{-1} \times \left[-(\omega^2 - \Omega_o{}^2) \pm \sqrt{\left(\frac{2I_oD_1\Omega_o{}^2}{3\,\omega} \right)^2 - \left(\frac{R_L + R_c}{L} \right)^2 \omega^2} \right] \right\}^{\frac{1}{2}} . \tag{115}$$

If ΔF is small, $\omega^2 \doteq \Omega_o{}^2$, so if I_o is considerably above the subharmonic threshold

$$I_c \doteq \left\{ \frac{6}{5D_1{}^2} \left(1 + \frac{9D_2}{10D_1{}^2} \right)^{-1} \left[-(\Omega_o + 2\pi\Delta F)^2 + \Omega_o{}^2 \pm \sqrt{\left(\frac{2I_oD_1\Omega_o}{3} \right)^2 - \left(\frac{R_c + R_L}{L} \right)^2 \Omega_o{}^2} \right] \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{6\Omega_o}{5D_1{}^2} \left(1 + \frac{9D_2}{10D_1^2} \right)^{-1} \left[-2\pi\Delta F - 2\pi^2 \frac{\Delta F^2}{\Omega_o} \pm \sqrt{\left(\frac{2I_oD_1}{3} \right)^2 - \left(\frac{R_c + R_L}{L} \right)^2} \right] \right\}^{\frac{1}{2}} .$$

For the ET61 sample No. 3 capacitor with the constants given above for its circuit and $I_o = 6.2$ ma. r.m.s., this equation reduces to

$$I_{c_{r.m.s.}} = \left\{ 39\,[0.2 I_{o_{r.m.s.}}{}^2 - 4]^{\frac{1}{2}} - 0.49(1 + 0.12 \times 10^{-3}\Delta F)\Delta F \right\}^{\frac{1}{2}}$$

$$= [75 - 0.49(1 + 0.12 \times 10^{-3}\Delta F)\Delta F]^{\frac{1}{2}} \text{ ma} ,$$

which is plotted along with the experimental tuning curve in Figure 17. The theoretical and experimental curves of Figure 17 agree satisfactorily for ΔF positive and over the range of ΔF for which the theoretical assumption holds. The theoretical and experimental curves do not agree near the lower frequency limit for the subharmonic. Theoretical tuning curves for the ET46 and K3300 capacitors are not given, since the effects of dielectric heating would have destroyed the correspondence of the experimental and theoretical curves. However, the limits of the tuning range over which the assumption of the theoretical work is valid--that is, the assumption that $\omega^2 - \Omega_o{}^2$ is of order ϵ^2 holds--are indicated by pairs of vertical lines on the ΔF scale of the experimental curves of Figure 19. These limits are found by normalizing the nonlinear differential equation. From equation (68) for $f(q) = D_1 q$ and $n = 2$

$$L \frac{d^2q}{dt^2} + (R_c + R_L) \frac{dq}{dt} + \frac{q}{C_o} (1 + D_1 q) = I_o |Z| \cos 2 \quad t \quad .$$

Or, if the change of variable $\theta = \omega t$ is made

$$\frac{d^2q}{d\theta^2} + \frac{(R_c + R_L)}{\omega L} \frac{dq}{d\theta} + \frac{\Omega_o{}^2 q}{\omega^2} (1 + D_1 q) = \frac{I_o |Z|}{\omega^2 L} \cos 2 \omega t \ .$$

Now let $q = Q_s x$ where x is of the order unity and Q_s is a scale factor. Then

$$\frac{d^2x}{d\theta^2} + \frac{(R_c + R_L)}{\omega L} \frac{dx}{d\theta} + \frac{\Omega_o{}^2}{\omega^2} x + \frac{\Omega_o{}^2}{\omega^2} D_1 Q_s x^2 = \frac{I_o |Z|}{\omega^2 L Q_s} \cos 2 \omega t.$$

Since x and $\frac{\Omega_o{}^2}{\omega^2}$ are of order unity, the coefficient $D_1 Q_s$ is a measure of the amount of nonlinearity of this equation. The experimental data above have been taken at a frequency of about 4 kilocycles with currents less than about 10 milliamperes. Therefore, the maximum value of charge observed in the experimental work is about 4×10^{-7} coulomb. Then, if Q_s is assumed to be 4×10^{-7}, x will be of order unity. Since D_1 is less than 10^6 for the dielectrics used at 4 kilocycle frequencies, the coefficient

$$\frac{\Omega_o^2}{\omega^2} \; D_1 Q_s$$

is less than 0.4 and the differential equation is almost linear. Since the loss factor of the tuned circuit

$$\frac{R_L + R_c}{\omega L}$$

was measured to be 0.08 or less, the losses are of the second order of smallness; that is, they are of order ϵ^2. Finally the detuning is of order ϵ^2 provided

$$D_1^2 Q_s^2 \geq \left|\frac{\Omega_o^2}{\omega^2} - 1\right| = \left|\frac{\Omega_o^2 - (\Omega_o + 2\pi\Delta F)^2}{\omega^2}\right| \doteq \left|\frac{4\pi\Delta F}{\omega_o}\right| = \left|\frac{2\Delta F}{f_o}\right| .$$

The fractional tuning range, over which the analytical assumption that the detuning is of order ϵ^2 is valid is given approximately by

$$-\frac{D_1^2 Q_s^2}{2} \leq \frac{\Delta F}{f_o} \leq \frac{D_1^2 Q_s^2}{2} . \tag{116}$$

Thus the limits on the valid tuning range, for $D_1 = 0.5 \times 10^6$ and $Q_s = 0.4 \times 10^{-7}$, are given approximately by $\left|\frac{\Delta F}{f_o}\right|$ less than 2 percent or $\Delta F \doteq \pm$ 80 cycles for a frequency f_o of 4 kilocycles.

A comparison of the computed and measured subharmonic capacitor current shows that the measured value is appreciably less than that calculated but that the subharmonic threshold is approximately the same. In order to determine how heating affected the above results, Sample No. 4 of ET61 dielectric was soldered to a thin sheet of copper which was then bolted to a block of aluminum to serve as a heat radiator. Table 3 gives the data on this element measured by harmonic voltage analysis with approximately a sine wave of charge. The operating conditions of the 6SJ7 pentode current source were changed to a screen voltage of 125 volts and a grid bias of - 3 volts. The reduction of screen potential increased the tube plate resistance and thus caused lower tank losses. The voltage divider, which in Figure 15 was used to measure v, was omitted to retain as low circuit losses as possible. This capacitor unit was used in the tank circuit of the 6SJ7. The frequency of the driving source was adjusted to the frequency at which minimum voltage was required to excite the subharmonic. One half this frequency for minimum voltage was considered the reference frequency and detuning was measured from this value. The amplitude of subharmonic capacitor current is plotted in Figure 20 as a function of tube current for each excitation frequency; also shown is a theoretical curve computed from equation (115) for zero detuning, $D_1 = 0.48 \times 10^6$, $D_2 = 0.16 \times 10^{12}$, and a tank Q of 28. These polynomial coefficients are given in Table 3 for the ET61 sample No. 4 with a 250 volt bias. This theoretical curve agrees quite well with the experimental $\Delta F = 0$ curve, except in the region of the subharmonic threshold. The D_1 and D_2 coefficients and the circuit loss vary considerably for small capacitor currents, so the errors in the computed curves near the subharmonic threshold probably arise from errors in the measured values of D_1 and circuit loss resistance. D_1, D_2 and R_c were assumed to be constant in the theoretical analysis.

It thus appears that tuning errors and errors in the determination of the nonlinear characteristic polynomial approximation are greater than any errors in the analytical treatment of the subharmonic amplitude response. This is not true for the response as a function of tuning. The

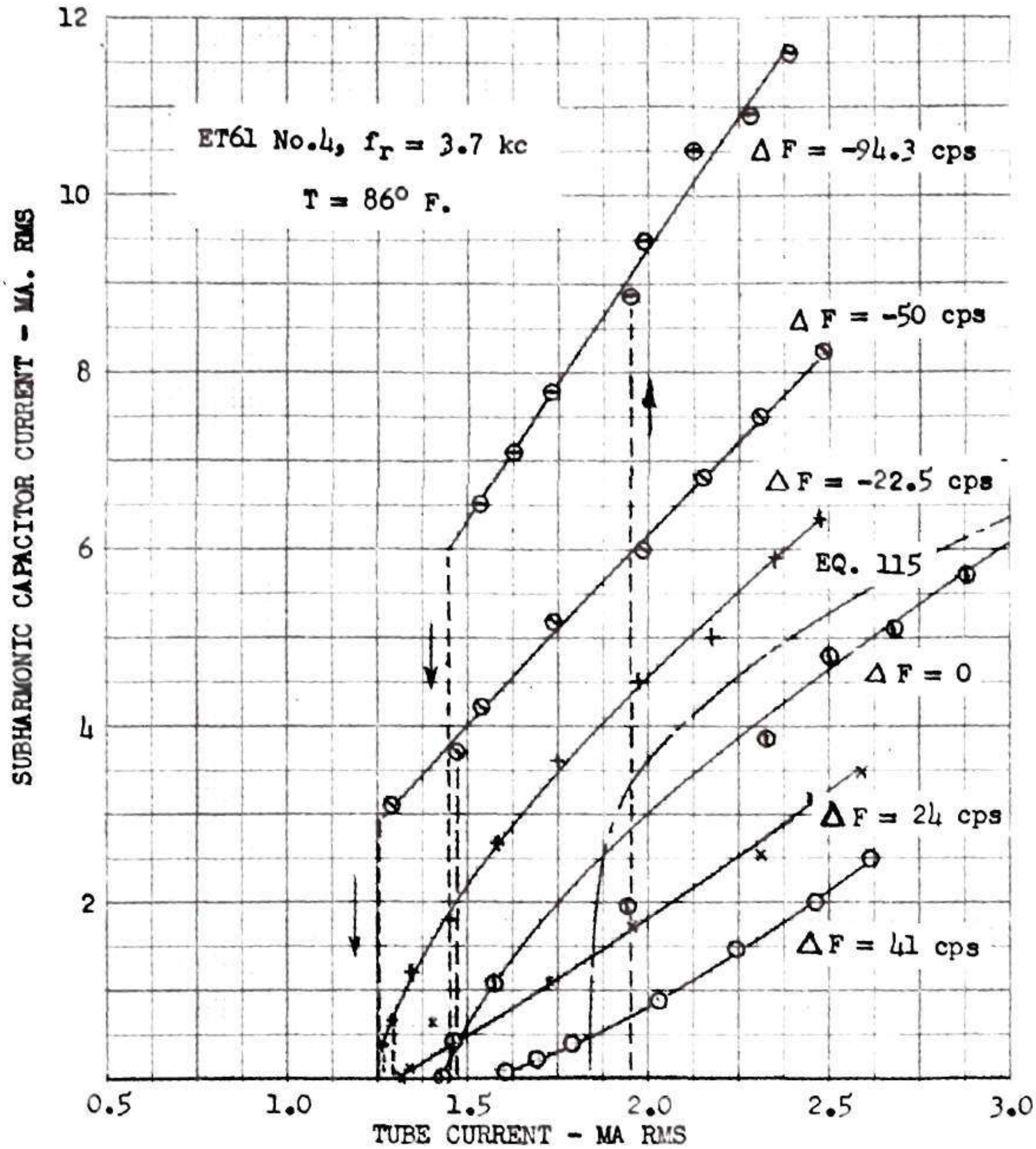
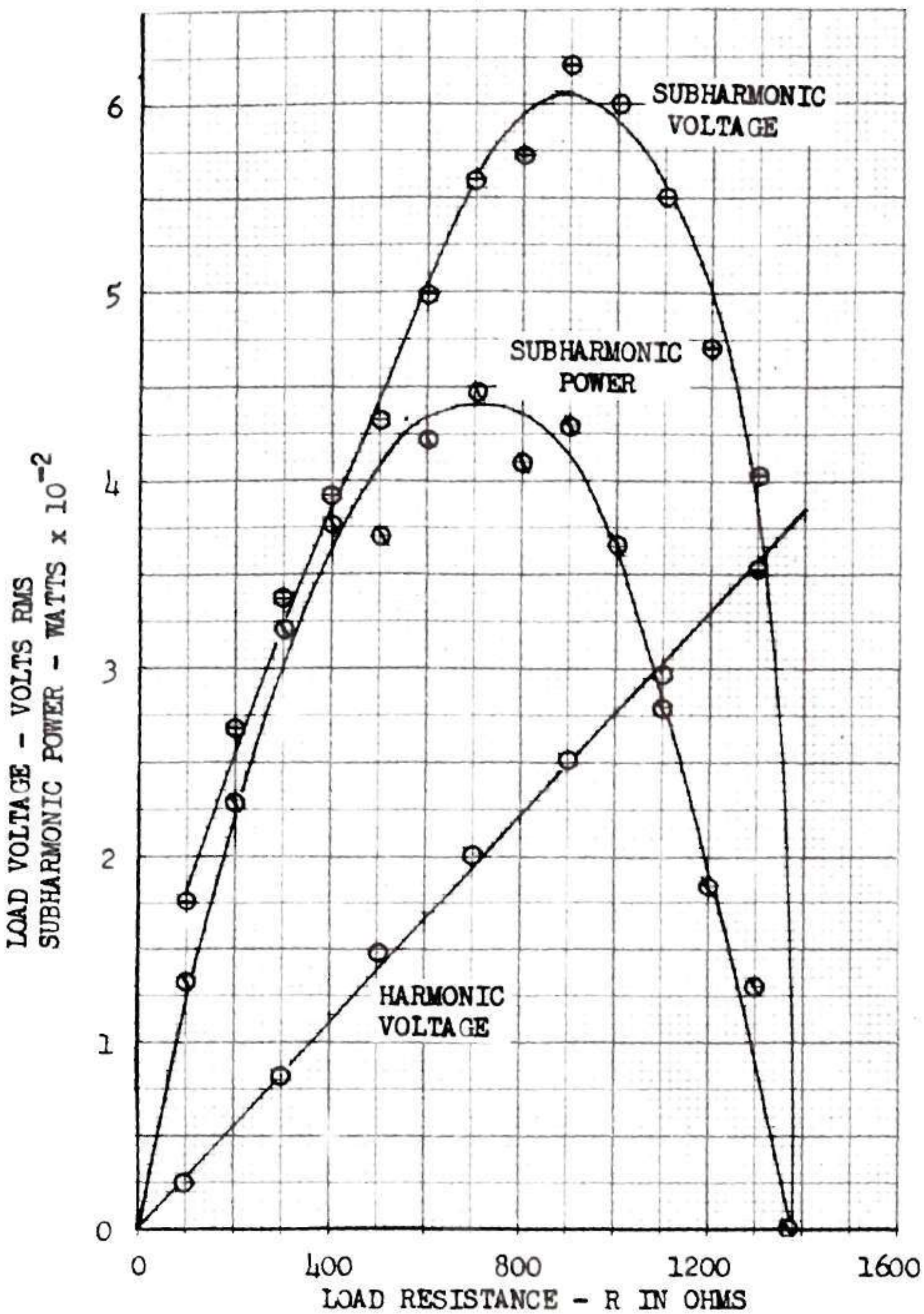


FIG. 20. EFFECT OF TUNING ON SECOND SUBHARMONIC RESPONSE

analytical treatment does correctly predict the hysteresis effect in initiating the subharmonic if $f < f_o$ but it does not yield an analytical lower limit to the frequency at which the subharmonic may exist. However, the low frequency limit on the subharmonic is outside the range of ΔF for which the detuning is of order ϵ^2 as can be seen from Figures 17 and 19. The measured and computed tuning curves of Figure 17 have the same shape except near the lower frequency limit of the subharmonic response. These curves are displaced by the amount that the reference frequency differs from the true resonant frequency. It seems that any errors resulting from analytical approximation are masked in experimental work due to variations of the nonlinear characteristic with amplitude and errors in tuning.

Another capacitor with ET61 dielectric sample No. 2 soldered to a copper rod was used to investigate the variation of subharmonic current and power output with tank or load resistance. The data were taken with the 6SJ7 operating as a current source with 250-volt screen voltage, grid No. 1 bias of - 5 volts, tank inductance of 0.66 henries, and a reference frequency of 4 kilocycles. The audio-frequency grid voltage was maintained constant at 3.8 volts r.m.s., or a tube current of 8.6 ma., and at twice the reference frequency, or 8 kilocycles. A variable resistor R was placed in series with the tank coil and the voltage across this resistor was measured for various values of this load resistance. Curves of voltage variation and power are plotted as a function of R in Figure 21.

A series of typical waveforms and Lissajous patterns for various experimental conditions on the second order subharmonic, with the circuit of Figure 15, is given in Figure 22. These waveforms were obtained by



ET61 No.2, $\Delta F = 0$
$e_s = 3.8$ volts

FIG. 21. LOAD VOLAGE AND POWER VERSUS LOAD RESISTANCE

photographing the trace of a Dumont 304H oscilloscope with a 35 millimeter oscilloscope camera. Those traces showing charge were obtained by taking the voltage across a large linear capacitor in series with the nonlinear capacitor of Figure 15.

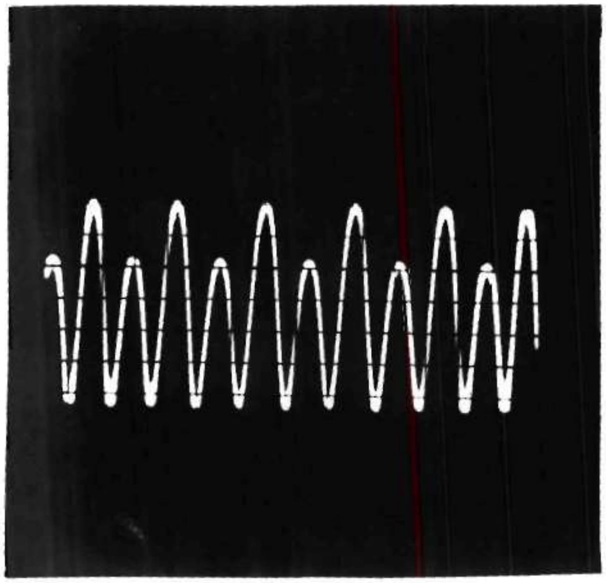
In this section measurements have shown that the circuit losses do vary with amplitude of the response: that is, the resistance voltage drop is not a linear function of current. No attempt has been made to include this in the analytical treatments since the losses were assumed of order ϵ^2 and their variation is of still smaller order. In some nonlinear problems first order losses may occur and the variations of circuit loss could be significant. These problems could be analyzed by the successive approximation method of Chapter II by letting the loss term be

$$R(q)\frac{dq}{dt} = R_o(1 + r_1q + r_2q + r_2q^2 + - - -)\frac{dq}{dt} \quad .$$

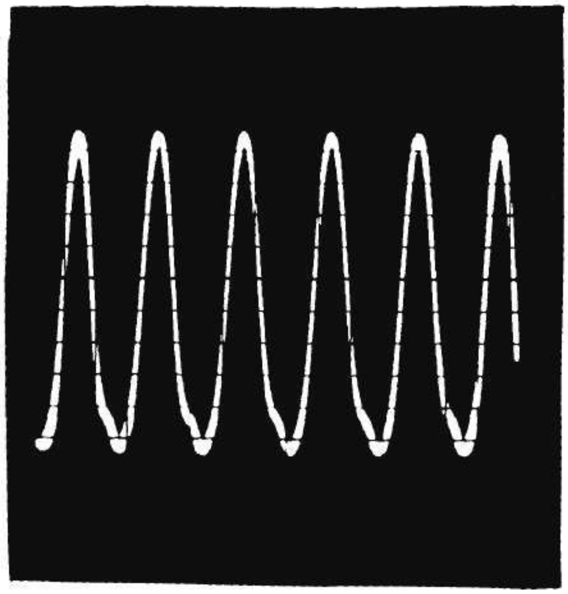
In the second order subharmonic case only the linear R_o and $r_2R_oq^2$ terms are likely to be significant. It is also possible to develop an analysis of the second order subharmonic, when the detuning is of order ϵ. This has not been carried out herein since it offers no major new information and is not justified in view of experimental errors.

It is interesting to consider the upper frequency limits at which the second subharmonic can be obtained in single loop circuits with available dielectrics. From equation (104) the condition that the subharmonic exist, for $\omega = \Omega_o$, is

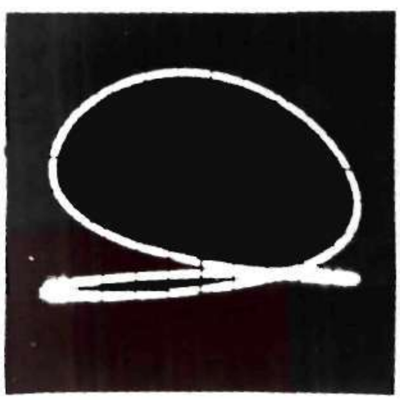
$$\frac{2I_o \omega D_1}{3} > \frac{(R_c + R_L)}{L} \omega$$



CAPACITOR CURRENT

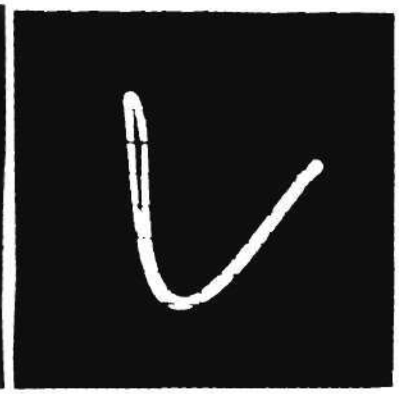


CAPACITOR VOLTAGE



LISSAJOUS FIGURE OF CAPACITOR VOLTAGE AND GRID VOLTAGE

LISSAJOUS FIGURE OF CAPACITOR CURRENT AND GRID VOLTAGE



LISSAJOUS FIGURE OF CAPACITOR CURRENT AND VOLTAGE

FIG. 22. SECOND SUBHARMONIC WAVEFORMS FOR CIRCUIT OF FIGURE 15.

Or

$$I_o \geqq \frac{3}{2} \frac{(R_c + R_L)}{\omega L D_1} \omega = \frac{3\omega}{2D_1 Q} = \frac{3\pi f}{D_1 Q} \quad . \tag{117}$$

In Chapter III it was shown that the coefficient D_1 varies inversely with the capacitor plate area or inversely as the coefficient C_o. That is,

$$\frac{D_1'}{D_1} = \frac{C_o}{C_o'} \quad .$$

Thus, since the ET61 No. 3 capacitor has a D_1 coefficient of about 0.5 x 10^6 with a linear capacity C_o of 3000 μμf, a capacitor of 300 μμf linear capacity would have a D_1 of approximately 5 x 10^6. Measurements of the capacitor Q indicate that these dielectrics retain moderate Q at frequencies up to the order of 20 megacycles or higher. Thus, if a circuit Q of 20 can be maintained and C_o is 300 μμf (about the smallest ET61 unit readily handled so that D_1 is 5 x 10^6, the frequency limit is

$$f < \frac{D_1 Q}{3\pi} I_o = \frac{5 \times 10^6 \times 20}{3\pi} I_o \quad .$$

Accordingly, if an r.m.s. current of 10 ma from a high-plate-resistance pentode is attainable, the frequency for the single-loop second-order subharmonic is limited to less than about 150 kilocycles for the subharmonic frequency or a driving frequency of 300 kilocycles.

An experimental check on this upper frequency limit was made using a 6AK5 tube with a plate voltage of 250 volts, an ET61 capacitor and a tank inductance of 500 μh. at a forcing frequency, 2 ω, of 820 kc. The 410 kc subharmonic observed in this test was the highest subharmonic frequency observed in a single loop circuit. The tank possessed a Q of about 40 and the required tube current was approximately 16 milliamperes r.m.s. $D_1 \doteq 3.6 \times 10^6$ and $C_o \doteq 300$ μμf, as measured from a 60-cycle hysteresis loop. The theoretical upper frequency limit of 346 kc is in reasonable agreement with the measured limit of 410 kc.

It is desirable before leaving the relatively simple single-loop second-subharmonic to consider how the subharmonic will vary if the driving current amplitude is modulated. Physically, it is clear that if the driving current I_o is varied slowly the subharmonic amplitude will follow these variations smoothly, but not linearly, provided I_o remains above the threshold condition for the subharmonic. Unfortunately, the first order differential equations (94) and (95) which define the amplitude and phase of the solution are themselves nonlinear and the variables are not separable. Thus the time variation of the subharmonic amplitude is not readily obtainable analytically. If the driving current is constant, it is possible to determine graphically how the solution approaches its final values by plotting the variation of $\frac{da}{d\phi}$. This has been considered by Reuter[38] for the case of zero third order curvature. An estimate of the variation of the subharmonic amplitude with time is obtainable from the perturbed equations (96), (97) and (99), which were used to determine the stability of the solution. If the stable subharmonic exists and is perturbed from its equilibrium condition, it will return to this equilibrium in accordance with the function $e^{p_1 t} + e^{p_2 t}$, where p_1, p_2 are given by equation (99). Thus p_1 of (99) serves to define a type of time constant for small variations of the subharmonic. In terms of the circuit constants (99) is

$$p_1,\ p_2 = \left(\frac{R_L + R_c}{4L}\right)\left\{-1 \pm \left[1 - \frac{20L^2 a_o^{\ 2}}{9\omega(R_L + R_c)^2} \times \sqrt{\left(\frac{I_o|Z|D_1\Omega_o^{\ 2}}{L}\right)^2 - \frac{3\omega^{6}(R_L + R_c)^2}{L^2}}\ \right]^{\frac{1}{2}}\right\}$$

where

$$a_o^{\ 2} = \frac{6\omega^2}{5D_1\Omega_o^{\ 2}}\left[-\left(\omega^2 - \Omega_o^{\ 2}\right) \pm \sqrt{\left(\frac{I_o|Z|D_1\Omega_o^{\ 2}}{3\omega^2 L}\right)^2 - \left(\frac{R_L + R_c}{L}\right)^2 \omega^2}\ \right].$$

From these equations it is seen that the effective time constants for the subharmonic amplitude will vary from $\frac{1}{2}$ to 1 and 0 to $\frac{1}{2}$ of the linear time constant $\frac{2\ L}{R_L + R_c}$ as the driving current varies from the subharmonic threshold to large values. Thus it appears reasonable that the subharmonic should follow variations in driving current, provided this current amplitude does not vary at a faster rate than $\frac{1}{2}$ the reciprocal of the time constant

$$\frac{2\ L}{R_L + R_c} = \frac{2Q}{\Omega_o} ,$$

and the driving current remains well above the threshold.

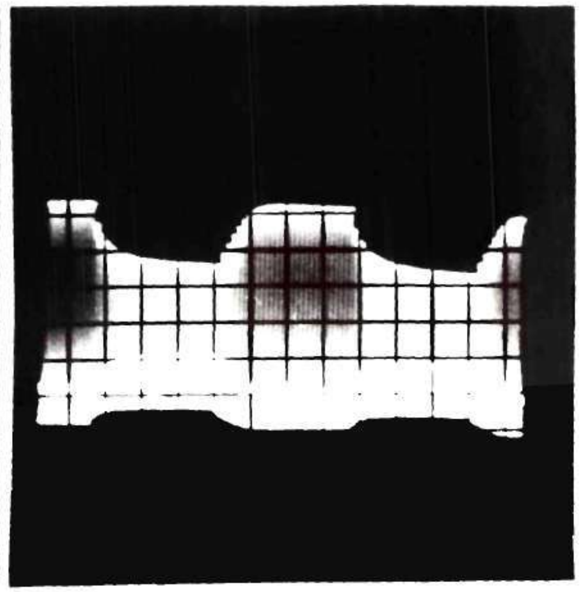
Experimental data on the transient behavior of the subharmonic solution were obtained by applying a square wave gate to the suppressor grid of the 6SJ7 current source. Oscilloscope pictures of the tank voltage waveforms, including the subharmonics for different square wave voltages and frequencies, are given in Figure 23 for zero detuning. These pictures show that the subharmonic has a rise time from zero to 0.632 of its final value of about 2.6 milliseconds. The tank circuit used had a Q of about 10, an inductance of 0.66 henries and a resonant frequency of 4.0 kilocycles with the ET61 No. 2 capacitor. The linear time constant is about 5 milliseconds.

Figure 24 gives a series of oscilloscope pictures showing waveforms of the envelope of the tank voltage, when the driving current is sine wave amplitude modulated. These waveforms were taken with a carrier frequency of 90 kilocycles, which was approximately the tank resonant frequency, for the various indicated values of amplitude modulation index and modulating frequencies. The carrier voltage was 2.8 volts and the nonlinear capacitor was an ET61 unit of approximately 300 micro-microfarads capacity. A 2.5-volt carrier voltage was necessary to sustain the subharmonic without modulation. The waveforms of Figure 24 show that the subharmonic tends to zero in the modulation troughs, if the modulation index is 0.1 with 50 cycle modulation. The subharmonic envelope is a distorted sine wave for a modulation index of 0.05 with a modulating frequency of 50 cycles. At a modulation frequency of 200 or 2000 cycles the degree of modulation, for which the subharmonic exists, is increased; also the subharmonic envelopes are more nearly sinusoidal. Thus as the modulation frequency increases less envelope distortion occurs in the subharmonic response to an amplitude modulated wave.

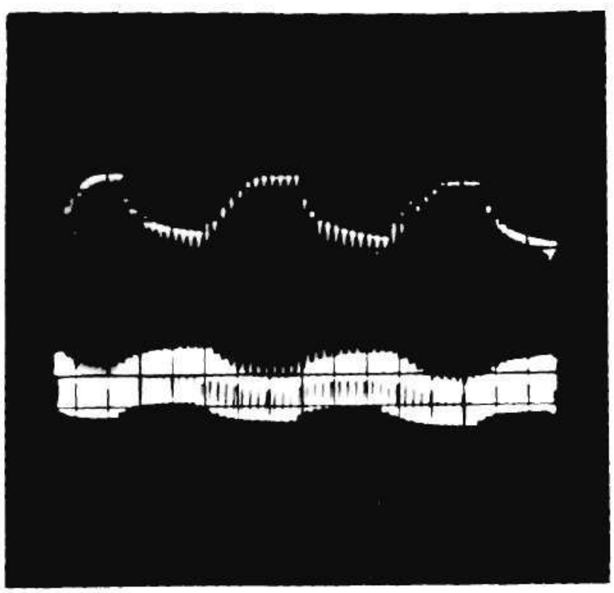
Third and Higher Order Subharmonics.

This section treats the properties of higher order subharmonics in singly resonant circuits. In particular, it is shown that subharmonics of

75 CYCLE MODULATION

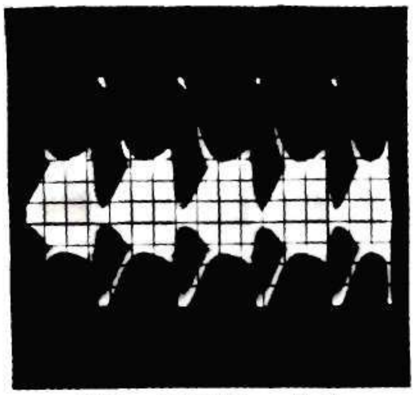


75 CYCLE MODULATION

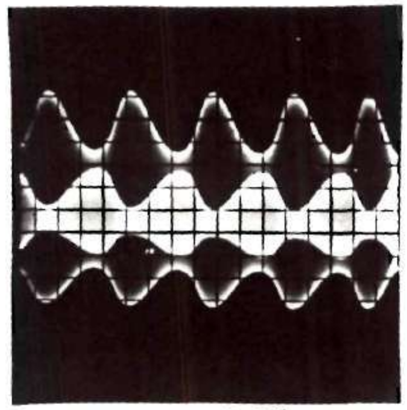


180 CYCLE MODULATION

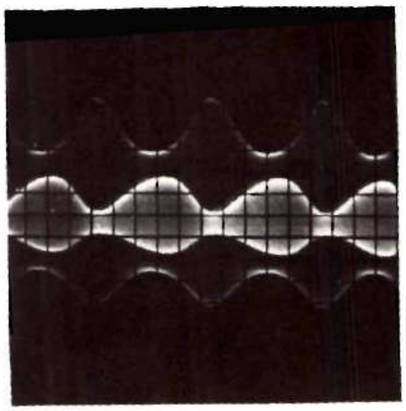
FIG. 23. SECOND SUBHARMONIC RESPONSE WITH SQUARE WAVE MODULATION



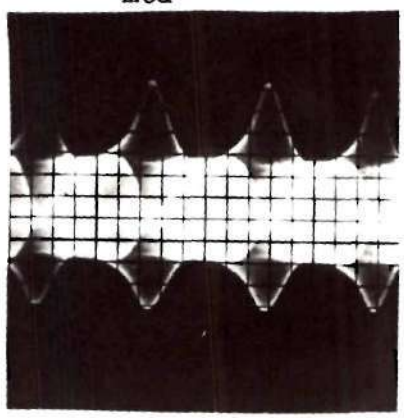
MOD. INDEX = 0.1
$f_{mod.}$ = 50 cps



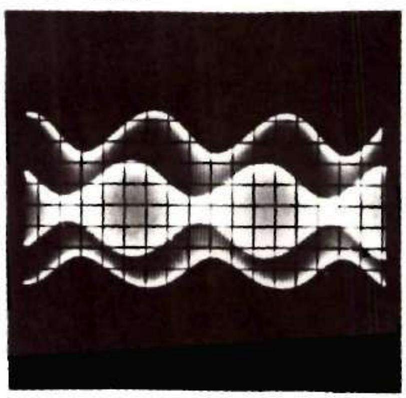
m = 0.05
f_{mod} = 50 cps



m = 0.1
f_{mod} = 200 cps



m = 0.2
f_{mod} = 200 cps



m = 0.4
f_{mod} = 2000 cps

m = 0.4
f_{mod} = 10,000 cps

FIG. 24. SECOND SUBHARMONIC RESPONSE WITH SINE WAVE AMPLITUDE MODULATION

greater than the second order will not build up from rest. Starting conditions for the third and fifth order subharmonics in saturable reactor circuits have been investigated experimentally by McCrumm[4] and McKune[59].

Higher order subharmonics and their initiation--Consider the nonlinear differential equation

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C_o}(1 + D_{n-1}q^{n-1}) = V \cos n\omega t . \qquad (118)$$

If $\theta = \omega t$, and $\frac{1}{LC_o} = \Omega_o^2$, this equation can be written as

$$\frac{d^2q}{d\theta^2} + \frac{R}{\omega L}\frac{dq}{d\theta} + \frac{\Omega_o^2 q}{\omega^2}(1 + D_{n-1}q^{n-1}) = \frac{V}{\omega^2} \cos n\theta . \qquad (119)$$

If

$$\frac{R}{L} = \epsilon^2 k,\ \omega^2 - \Omega_o^2 = \epsilon^2 h,\ \frac{V}{\omega^2} = \epsilon F,\ D_{n-1} = \epsilon d_{n-1}$$

and all other nonlinear coefficients are neglected, equation (119) reduces to

$$\frac{dq^2}{d\theta^2} + q = \epsilon F \cos n\theta - \frac{\epsilon^2 k}{\omega}\frac{dq}{d\theta} + \frac{\epsilon^2 hq}{2} - \frac{\epsilon\Omega_o^2}{2} d_{n-1}q^n . \qquad (120)$$

Equation (29) of the Kryloff approximation analysis yields a zeroth approximation to the solution of the form $a\sin(\theta + \phi)$. By equation (30), the first approximation is obtained from

$$\frac{\partial^2 Z_1}{\partial \theta^2} + Z_1 = F \cos n\theta + \frac{2\sigma_1 a}{\omega}\sin(\theta + \phi) - \frac{2A_1}{\omega}\cos(\theta + \phi) \qquad (121)$$

$$-\frac{\Omega_o^{\,2}}{\omega^2} d_{n-1}\, a^n \sin^n(\theta + \phi) = F\cos n\theta + \frac{2\sigma_1 a}{\omega}\sin(\theta + \phi)$$
$$-\frac{2A_1}{\omega}\cos(\theta + \phi) - \frac{\Omega_o^{\,2}}{\omega^2} d_{n-1}\, a^n \left[\frac{M_o}{2} + \sum_{k=1}^{n} M_k \cos k(\theta + \phi) + N_k \sin k(\theta + \phi)\right] ,$$

where $N_k = 0$ and

$$M_k = \frac{2}{\pi}\int_0^{2\pi} \sin^n(\theta + \phi)\cos k(\theta + \phi)\, d\,(\theta + \phi) \tag{122}$$
$$= \frac{1}{2^{n-1}} \cos k\frac{\pi}{2} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+k}{2} + 1\right)\Gamma\left(\frac{n-k}{2} + 1\right)},$$

if n is even; or $M_k = 0 = M_o$, and

$$N_k = \frac{\sin k\,(\frac{\pi}{2})}{2^{n-1}} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+k}{2} + 1\right)\Gamma\left(\frac{n-k}{2} + 1\right)} \tag{123}$$

if n is odd. In order that Z_1 contain no secular terms, the terms of period 2π in θ must be zero, so

$$A_1 = 0$$

and

$$\frac{2\sigma_1 a}{\omega} = \frac{\Omega_o^{\,2}}{\omega^2} d_{n-1} \frac{a^n}{2^{n-1}} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+1}{2} + 1\right)\Gamma\left(\frac{n-1}{2} + 1\right)} \tag{124}$$

if n is odd. Or, $A_1 = \sigma_1 = 0$ if n is even (125)

and

$$Z_1 = \frac{F}{1 - n^2} \cos n\theta - \frac{\Omega_o^{\ 2} d_{n-1} a^n}{\omega^2} \left[\frac{M_o}{2} + \sum_{k=2}^{n} \frac{M_k}{1 - k^2} \cos k(\theta + \phi) \right]. \qquad (126)$$

Then, if n is odd

$$Z_1 = \frac{F}{1 - n^2} \cos n\theta - \frac{\Omega_o^{\ 2} d_{n-1} a^n}{\omega^2} \sum_{k = 2}^{n} \frac{N_k}{1 - k^2} \sin k(\theta + \phi) \qquad (127)$$

Since $A_1 = 0$ for all n, equation (36), which defines the solution to a second approximation, becomes

$$\frac{\partial^2 Z_2}{\partial \theta^2} + Z_2 = \frac{2\sigma_2 a^n}{\omega} \sin(\theta + \phi) - \frac{2\sigma_1}{\omega} \frac{\partial^2 Z_1}{\partial \phi \partial \theta} - \frac{2A_2}{\omega} \cos(\theta + \phi) \qquad (128)$$

$$+ \frac{\sigma_1^{\ 2} a}{\omega^2} \sin(\theta + \phi) - \frac{\sigma_1 a}{\omega^2} \frac{\partial \sigma_1}{\partial a} \cos(\theta + \phi)$$

$$- n \frac{\Omega_o^{\ 2}}{\omega^2} d_{n-1}\, a^{n-1}\, Z_1 \sin^{n-1}(\theta + \phi)$$

$$- \frac{ka}{\omega} \cos(\theta + \phi) + \frac{ha}{\omega^2} \sin(\theta + \phi) .$$

Now

$$\sin^{n-1}(\theta + \phi) = \frac{P_o}{2} + \sum_{k = 1}^{n - 1} P_k \cos k(\theta + \phi) + S_k \sin k(\theta + \phi),$$

where

$$P_k = \frac{1}{2^{n - 2}} \cos k \frac{\pi}{2} \frac{\Gamma(n)}{\Gamma\left(\frac{n - 1 + k}{2} + 1\right)\Gamma\left(\frac{n - 1 - k}{2} + 1\right)} \qquad (129)$$

$$S_k = 0$$

or

$$P_k = 0,$$

$$S_k = \frac{\sin k \frac{\pi}{2}}{2^{n-2}} \frac{\Gamma(n)}{\Gamma\left(\frac{n-1+k}{2}+1\right)\Gamma\left(\frac{n-1-k}{2}+1\right)}, \tag{130}$$

if n is even.

Thus

$$Z_1 \sin^{n-1}(\theta+\phi) = \left\{ \frac{F\cos n\theta}{1-n^2} - \frac{\Omega_o{}^2 d_{n-1} a^n}{\omega^2} \left[\frac{M_o}{2} + \sum_{k=2}^{n} \frac{M_k}{1-k^2} \cos k(\theta+\phi) \right] \times \sum_{k=1}^{n-1} S_k \sin k(\theta+\phi) \right\}$$

if n is even,

or

$$Z_1 \sin^{n-1}(\theta+\phi) =$$

$$\left\{ \frac{F\cos n\theta}{1-n^2} - \frac{\Omega_o{}^2 d_{n-1} a^n}{\omega^2} \sum_{k=2}^{n} \frac{N_k}{1-k^2} \sin k(\theta+\phi) \right\}$$

$$\times \left[\frac{P_o}{2} + \sum_{k=1}^{n-1} P_k \cos k(\theta+\phi) \right] \quad \text{if n is odd.}$$

It is necessary only to know the terms of period 2π in θ to derive a second approximation for the amplitude and phase of the n-th. subharmonic. Thus Z_2 need not be solved for and only the terms of period 2π are needed in the expansion of $Z_1 \sin^{n-1}(\theta + \phi)$. From the recurrence formula for M_k and N_k only even cosine harmonics and odd sine harmonics appear. So the terms of period 2π in $Z_1 \sin^{n-1}(\theta + \phi)$, for n even, are given by

$$- \frac{FS_{n-1}}{2(1-n^2)} \sin[\theta - (n-1)\phi] - \frac{\Omega_o{}^2 d_{n-1} a^n}{2\omega^2} \Big[M_o S_1 \qquad (131)$$

$$+ (S_3 - S_1) \frac{M_2}{1-2^2} + (S_5 - S_3) \frac{M_4}{1-4^2}$$

$$+ \text{---} (S_{n-1} - S_{n-3}) \frac{M_{n-2}}{1-(n-2)^2} - \frac{M_n}{1-n^2} S_{n-1} \Big] \sin(\theta + \phi)$$

$$= - \frac{FS_{n-1}}{2(1-n^2)} \sin[\theta - (n-1)\phi] - \frac{\Omega_o{}^2 d_{n-1} a^n}{2\omega^2} K_{SM} \sin(\theta + \phi),$$

where

$$K_{SM} = M_o S_1 + (S_3 - S_1) \frac{M_2}{1-2^2} + \text{---} + (S_{n-1} - S_{n-3}) \frac{M_{n-2}}{1-(n-2)^2}$$

$$- \frac{M_n}{1-n^2} S_{n-1} \quad .$$

For n odd, the terms of period 2π in $Z_1 \sin^{n-1}(\theta + \phi)$ are

$$\frac{FP_{n-1}}{2(1-n^2)} \cos[\theta - (n-1)\phi] - \frac{\Omega_o{}^2 d_{n-1} a^n}{2\omega^2} \Big[\frac{N_3}{1-3^2} (P_2 - P_4) \qquad (132)$$

$$+ \frac{N_5}{1-5^2} (P_4 - P_6) + \text{---} + \frac{N_{n-2}}{1-(n-2)^2} (P_{n-1} - P_{n-3})$$

$$+ \frac{N_n}{1-n^2} (P_{n-1} \Big] \sin(\theta + \phi) = \frac{FP_{n-1}}{2(1-n^2)} \cos[\theta - (n-1)\phi]$$

$$- \frac{\Omega_o{}^2 d_{n-1} a^n}{2\omega^2} K_{PN} \sin(\theta + \phi).$$

It is possible to express the coefficients K_{SM} and K_{PM} in terms of the Gamma functions or factorials by the use of their recurrence formulae.

Since Z_1 contains no terms of period 2π, the conditions that Z_2 contain no secular terms are, for n even,

$$\frac{2\sigma_2 a}{\omega} + \frac{\sigma_1^{\,2} a}{\omega^2} + \frac{ha}{\omega^2} + \frac{nFS_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-1}}{2(1-n^2)\omega^2}\cos n\phi + \frac{n\Omega_o^{\,4} d_{n-1}^2 a^{2n-1}}{2\omega^4} KS_M = 0$$

$$-2\frac{A_2}{\omega} - \frac{\sigma_1 a}{\omega^2}\frac{\partial \sigma_1}{\partial a} - \frac{ka}{\omega} - \frac{nFS_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-1}}{2\omega^2(1-n^2)}\sin n\phi = 0$$

and, for n odd,

$$0 = 2\frac{\sigma_2 a}{\omega} + \frac{\sigma_1^{\,2} a}{\omega^2} + \frac{ha}{\omega^2} - \frac{nFP_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-1}}{2\omega^2(1-n^2)}\sin n\phi + \frac{n\Omega_o^{\,4} d_{n-1}^2 a^{2n-1}}{2\omega^4} K_{PN}$$

$$0 = -2\frac{A_2}{\omega} - \frac{\sigma_1 a}{\omega^2}\frac{\partial \sigma_1}{\partial a} - \frac{ka}{\omega} - \frac{nFP_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-1}}{2\omega^2(1-n^2)}\cos n\phi .$$

After substituting the value of σ_1 found in equations (124) and (125), it is found that, for n even,

$$\sigma_2 = \frac{\omega}{2}\left(-\frac{h}{\omega^2} - \frac{nFS_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-2}}{2(1-n^2)\omega^2}\cos n\phi - \frac{n\Omega_o^{\,4} d_{n-1}^2 a^{2\,n-2}}{2\omega^4} K_{SM}\right) \quad (133)$$

$$A_2 = \frac{\omega a}{2}\left(-\frac{k}{\omega} - \frac{nFS_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-2}}{2\omega^2(1-n^2)}\sin n\phi\right), \quad (134)$$

and, for n odd,

$$\sigma_2 = \frac{\omega}{2}\left[- \frac{(n-1)^2 \Omega_o^{\,4} d_{n-1}^2 a^{2\,n-2}}{4\omega^2} N_1^2 - \frac{h}{\omega^2} \right. \tag{135}$$

$$\left. + \frac{nFP_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-2}}{2\omega^2(1-n^2)} \sin n\,\phi - \frac{n\Omega_o^{\,4} d_{n-1}^2 a^{2\,n-2}}{2\omega^4} K_{PN} \right]$$

$$A_2 = \frac{\omega a}{2}\left[\frac{(n-1)\Omega_o^{\,4} d_{n-1}^2 a^{2\,n-3}}{4\omega^2} N_1^2 - \frac{k}{\omega} - \frac{nFP_{n-1}\Omega_o^{\,2} d_{n-1} a^{n-2}}{2\omega^2(1-n^2)} \cos n\phi \right]. \tag{136}$$

The second approximations to the amplitude and phase of the subharmonic of order n are the solutions of

$$\frac{da}{dt} = \epsilon A_1 + \epsilon^2 A_2 = 0$$

$$\frac{d\phi}{dt} = \epsilon \sigma_1 + \epsilon^2 \sigma_2 = 0 ,$$

where σ_1, A_1, σ_2 and A_2 are given by (124), (125), (133), (134), (135) and (136). The values of the subharmonic amplitude, a, and phase, ϕ, of the subharmonic of order n could theoretically be determined by examination of the equilibrium condition

$$\frac{da}{dt} = \frac{d\phi}{dt} = 0 . \tag{137}$$

However, (137) leads to a high order algebraic equation whose solution can be found only numerically for n greater than 2. The second approximation solution already presented for the second order subharmonic is a special case of the above n-th order subharmonic analysis.

The work of this Section has indicated the manner in which the second approximation to the subharmonic of order n may be calculated. Inspection of the equations for σ_1, σ_2, A_1 and A_2 shows that those terms which tend to sustain the subharmonic--that is, those terms containing the driving force which would make $\frac{da}{dt}$ positive--vanish as the $(n - 1)$ power of the subharmonic amplitude. Since the circuit resistance is constant, the dissipative terms are linear functions of the amplitude. It follows that, for $n > 2$, there is a least value of the subharmonic amplitude that can be sustained by a given excitation. Subharmonics of order greater than the second cannot build up from rest in singly resonant circuits but must initially be excited by some transient condition. This statement can be proven for large exciting currents and for the case of doubly resonant networks if one resonance is at the exciting frequency. Thus only the second order subharmonic, and its powers such as the fourth, eighth and sixteenth orders are self starting, since only these subharmonics can be made to build up smoothly from a quiescent condition. The fourth and eighth orders would be obtained as successive second order subharmonics.

Third order subharmonic--In this Section the third order subharmonic solution of a single loop circuit will be derived for the case in which the driving force is not small. This problem is treated to derive a result which can be compared to that obtained by Stoker[60] using Duffing's iteration method. It is assumed in this case that nonlinearity, loss and detuning are of the order of the small parameter ϵ, and that the driving voltage is of order unity. Under these assumptions the differential equation (118) can be written

$$\frac{d^2q}{dt^2} + \omega^2 q = \frac{V}{L}\cos 2\theta - \frac{R}{L}\frac{dq}{dt} - \Omega_o^{\,2} D_2 q^3 + q(\omega^2 - \Omega_o^{\,2}) \tag{138}$$

$$= \frac{V}{L}\cos 3\theta - \epsilon k\frac{dq}{dt} - \epsilon q^3 + \epsilon\, hq = \frac{V}{L}\cos 3\theta - \epsilon\left[k\frac{dq}{dt} + q^3 - hq\right].$$

Equation (138) is of the same form as (24). Hence, by the change of variable (25), equations (26) of the first approximation become

$$\dot{a} = -\frac{\epsilon}{\pi\omega}\int_0^{2\pi}\Big\{ ak\omega\cos(\omega t+\phi) + \frac{3Vk}{8\omega^2 L}\sin 3\omega t - ha\sin(\omega t+\phi) \tag{139}$$

$$+\frac{Vh}{8\omega^2 L}\cos 3\omega t + [a\sin(\omega t+\phi) - \frac{V}{8\omega^2 L}\cos 3\omega t]^3\Big\}\cos(\omega t+\phi)d(\omega t)$$

and

$$\dot{\phi} = \frac{\epsilon}{a\omega\pi}\int_0^{2\pi}\Big\{ ak\omega\cos(\omega t+\phi) + \frac{3Vk}{8\omega^2 L}\sin 3\omega t - ha\sin(\omega t+\phi) \tag{140}$$

$$+\frac{Vh}{8\omega^2 L}\cos 3\omega t + [a\sin(\omega t+\phi) - \frac{V}{8\omega^2 L}\cos 3\omega t]^3\Big\}\sin(\omega t+\phi)d(\omega t).$$

Thus

$$\dot{a} = -\frac{\epsilon}{\pi\omega}\Big\{ ak\,\omega + \int_0^{2\pi}[a\sin(\omega t+\phi) - \frac{V}{8\omega^2 L}\cos 3\,\omega t]^3 \times \tag{141}$$

$$\cos(\omega t+\phi)d(\omega t)\Big\} = -\frac{\epsilon}{\omega}\left[ak\omega + \frac{3a^2V}{32\omega^2 L}\cos 3\phi\right]$$

and

$$\dot{\phi} = \frac{\epsilon}{a\omega\pi}\Big\{ -ha\pi + \int_0^{2\pi}[a\sin(\omega t+\phi) - \frac{V}{8\omega^2 L}\cos 3\omega t]^3 \times \tag{142}$$

$$\sin(\omega t+\phi)d(\omega t)\Big\} = \frac{\epsilon}{a\omega}\left[-ha + \frac{3a^3}{4} + \frac{3a^2V}{32\omega^2 L}\sin 3\phi + \frac{3aV^2}{128\omega^4 L^2}\right]$$

The equilibrium points of $\dot{a} = \dot{\phi} = 0$, for $a \neq 0$, are given by

$$k\omega + \frac{3aV}{32\omega^2 L}\cos 3\phi = 0\ , \tag{143}$$

and

$$- h + \frac{3a^2}{4} + \frac{3aV}{32\omega^2 L} \sin 3\phi + \frac{3V^2}{128\omega^4 L^2} = 0 . \qquad (144)$$

Since

$$\sin 3\phi = [1 - \cos^2 3\phi]^{1|2} = \left[1 - \left(\frac{32\omega^3 k}{3aV}\right)^2\right]^{1|2} ,$$

the first approximation to the amplitude, a, of the subharmonic is a solution of the algebraic equation

$$- h + \frac{3a^2}{4} + \frac{3V^2}{128\omega^4 L} \pm \left[\left(\frac{3aV}{32\omega^2 L}\right)^2 - k^2\omega^2\right]^{1|2} = 0 \qquad (145)$$

In terms of the original circuit constants, the third order subharmonic amplitude is the solution of

$$- \frac{(\omega^2 - \Omega_o^2)}{D_2\Omega_o^2} + \frac{3a^2}{4} + \frac{3V^2}{128\omega^4 L^2} \pm \left[\left(\frac{3aV}{32\omega^2 L}\right)^2 - \left(\frac{R\omega}{LD_2\Omega_o^2}\right)^2\right]^{1|2} = 0. \qquad (146)$$

If the subharmonic is to exist, its amplitude must be such that

$$a \geq \frac{32\omega^2 L}{3V} \frac{R\omega}{LD_2\Omega_o^2} = \frac{32R\omega^3}{3VD_2\Omega_o^2} . \qquad (147)$$

Stoker[60] gives a similar condition which becomes, after changes to yield a consistent notation,

$$a \geq \frac{32R\Omega_o}{3VD_2} .$$

It is noted that the two results are identical if $\omega = \Omega_o$. The differences in the results obtained by the method of successive approximations and the methods of iteration and perturbation series are limited to terms which differ only in that generally $\omega \neq \Omega_o$. It is believed that this type of difference will always result due to the different techniques of treating frequency variations in the analytical methods. The discrepancy is always small, since it is assured

$$\left| \frac{\Omega_o^2}{\omega^2} - 1 \right| < \epsilon ,$$

so experimental results are unlikely to establish which is more nearly correct.

For purposes of comparison with the above solution a second approximation to the third subharmonic with driving voltage of order ϵ may be obtained by substituting $n = 3$ in the general results of the preceeding section on higher order subharmonics.

The third order subharmonic was not observed experimentally in single loop circuits with nonlinear dielectrics. Attempts to shock-excite this subharmonic failed. This result is not surprising since according to equation (147) a significant transient current at the subharmonic frequency must occur in order that the subharmonic be sustained with the small nonlinearities of the dielectrics used in this study; also the frequency must be such as to cause very little detuning.

CHAPTER V

SUBHARMONIC RESPONSE IN MULTIPLY RESONANT CIRCUITS

This Chapter treats analytically and experimentally subharmonic response in networks having two or more resonant frequencies. Emphasis is placed on networks in which subharmonics of third or higher order are self-starting.

Resonant Excitation of Second Order Subharmonics.

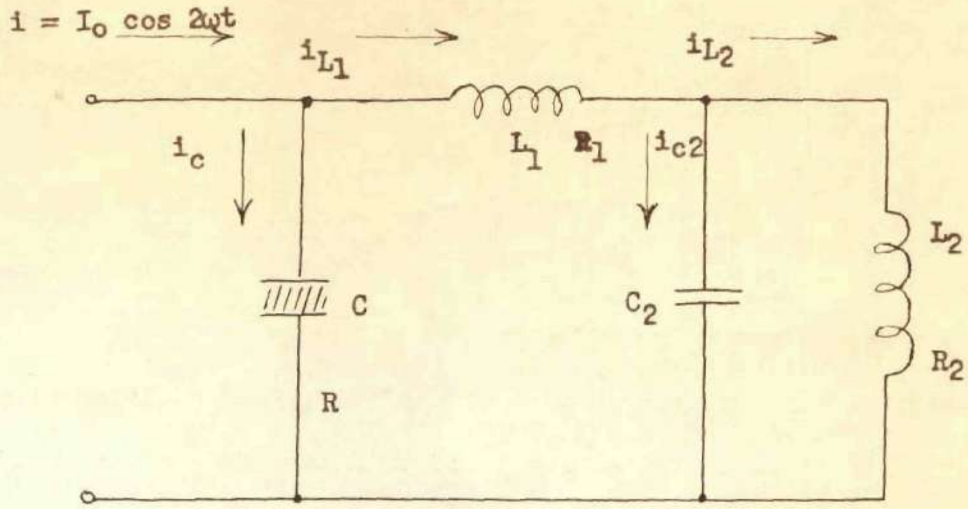
In this section the properties of networks having two degrees of freedom, one near the excitation frequency and the other at about one half that frequency, are investigated. The networks considered possess low losses and contain one nonlinear element. Figure 25 shows a voltage and a current-fed network of this class. It is assumed in each case that the elements L_1, L_2, C_1 and C_2 are selected so as to give the desired resonant conditions. C_1 is a nonlinear capacitor, and C_2 is a linear capacitor. The loss resistance of C_2 is assumed negligible.

If the capacitor C_1 is assumed to have the characteristic

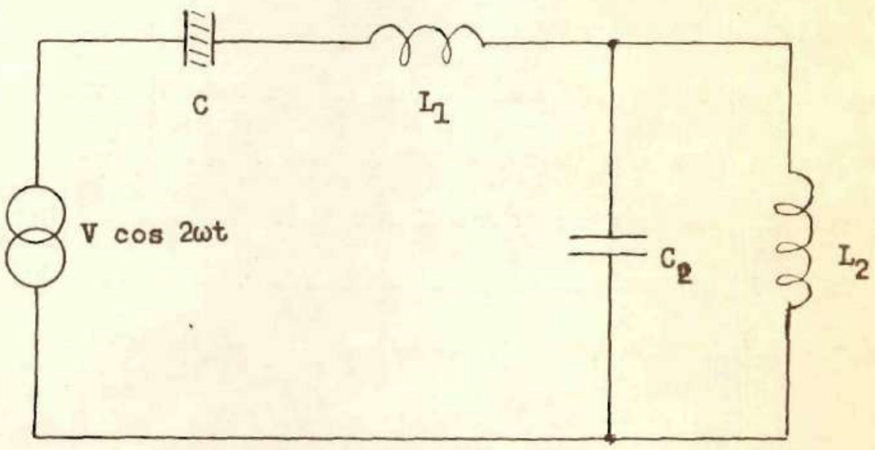
$$\frac{1}{C_1} = \frac{1}{C_o}(1 + Dq) ,$$

the Kryloff current and voltage equations for Figure 25 A are

$$i = i_c + i_{L1} = \frac{dq}{dt} + i_{L1} ,$$



A. CURRENT DRIVEN



B. VOLTAGE DRIVEN

FIG. 25. NETWORKS OF TWO DEGREES OF FREEDOM

$$i_{L1} = i_{L2} + i_{c2} = i_{L2} + \frac{dq_2}{dt} ,$$

$$\frac{q}{C_o} (1 + Dq) + R \frac{dq}{dt} = L_1 \frac{di_{L1}}{dt} + R_1 i_{L1} + \frac{q_2}{C_2} ,$$

and

$$\frac{q_2}{C_2} = i_{L2} R_2 + L_2 \frac{di_{L2}}{dt} .$$

These are reducible to the pair of second order nonlinear differential equations, with q and q_2 as the dependent variables, given by

$$L_1 \frac{d^2 q}{dt^2} + (R + R_1) \frac{dq}{dt} + \frac{q}{C_o} [1 + Dq] = L_1 \frac{di}{dt} + R_1 i + \frac{q_2}{C_2} ,$$

and

$$L_2 \frac{d^2 q_2}{dt^2} + R_2 \frac{dq_2}{dt} + \frac{q_2}{C_2} = R_2 (i - \frac{dq}{dt}) + L_2 \frac{d}{dt} (i - \frac{dq}{dt}) .$$

If it is assumed that the nonlinearity, losses, and exciting current are all of order ϵ, a small parameter, these can be written as

$$\frac{d^2 q}{dt^2} + \frac{q}{L_1 C_o} - \frac{q_2}{L_1 C_2} = \frac{di}{dt} + \frac{R_1}{L_1} i - \frac{(R + R_1)}{L_1} \frac{dq}{dt} - \frac{q^2 D}{L_1 C_o} = \tag{148}$$

$$\ddot{q} + \frac{q}{L_1 C_o} - \frac{q_2}{L_1 C_2}$$

$$= \epsilon f_1 (t, q_2, q, \dot{q})$$

and

$$\ddot{q} + \frac{q_2}{L_2C_2} + \ddot{q} = \frac{di}{dt} + \frac{R_2}{L_2} i - \frac{R_2\dot{q}}{L_2} - \frac{R_2\dot{q}_2}{L_2} = \epsilon f_2(t, q_2, \dot{q}_2, \dot{q}). \quad (149)$$

Following equation (40) of Chapter II, let

$$q = x_1 + x_2, \; q_2 = a_1x_1 + a_2x_2 \; , \quad (150)$$

where x_1 and x_2 are the components of q at the excitation frequency and its second subharmonic respectively. a_1 and a_2 are to be defined so as to reduce the differential equations to the normal form of equation (41). Then

$$\ddot{x}_1 + \ddot{x}_2 + \frac{x_1 + x_2}{L_1C_o} - \frac{a_1x_1 + a_2x_2}{L_1C_2} = \epsilon f_1 \quad (151)$$

$$(1 + a_1)\ddot{x}_1 + (1 + a_2)\ddot{x}_2 + \frac{a_1x_1}{L_2C_2} + \frac{a_2x_2}{L_2C_2} = \epsilon f_2 \; . \quad (152)$$

Equations (151) and (152) can be combined to yield

$$(a_2 - a_1)\ddot{x}_2 + \left[\frac{a_2}{L_2C_2} + \frac{a_2}{L_1C_2}(1 + a_1) - \frac{1 + a_1}{L_1C_o}\right] x_2 \quad (153)$$

$$+ \left[\frac{a_1}{L_2C_2} + \frac{a_1}{L_1C_2}(1 + a_1) - \frac{1 + a_1}{L_1C_o}\right] x_1 = \epsilon f_2 - \epsilon(1 + a_1)f_1,$$

and

$$(a_2 - a_1)\ddot{x}_1 + \left[- \frac{a_1}{L_2C_2} - \frac{a_1}{L_1C_2}(1 + a_2) + \frac{1 + a_2}{L_1C_o}\right] x_1 \quad (154)$$

$$+\left[-\frac{a_2}{L_2C_2}-\frac{a_2}{L_1C_2}(1+a_2)+\frac{1+a_2}{L_1C_o}\right]x_2=-\epsilon f_2+\epsilon(1+a_2)f_1.$$

If a_1 and a_2 are chosen as the two roots of

$$\frac{1+a}{L_1C_o}=a\left[\frac{1}{L_2C_2}+\frac{1+a}{L_1C_2}\right],$$

the above equations reduce to normal form and terms of order unity contain only x_2 in the first equation, only x_1 in the second. Thus let a_1 and a_2 be defined as

$$a_1=-\frac{1}{2}\left(1+\frac{L_1}{L_2}-\frac{C_2}{C_o}\right)-\frac{1}{2}\left[\left(1+\frac{L_1}{L_2}-\frac{C_2}{C_o}\right)^2+4\frac{C_2}{C_o}\right]^{1/2} \qquad (155)$$

$$a_2=-\frac{1}{2}\left(1+\frac{L_1}{L_2}-\frac{C_2}{C_o}\right)+\frac{1}{2}\left[\left(1+\frac{L_1}{L_2}-\frac{C_2}{C_o}\right)^2+4\frac{C_2}{C_o}\right]^{1/2} \qquad (156)$$

It is assumed that these roots are real and distinct. This is a necessary condition, if the circuit is to possess two distinct resonant frequencies. With a_1 and a_2 defined by (155) and (156), the differential equations for x_1 and x_2 can be written

$$\ddot{x}_1+\Omega_1^2\ x_1=\epsilon\frac{(1+a_2)}{a_2-a_1}f_1-\frac{\epsilon f_2}{a_2-a_1}$$

and

$$\ddot{x}_2+\Omega_2^2\ x_2=\frac{\epsilon f_2}{a_2-a_1}-\frac{\epsilon(1+a_1)}{a_2-a_1}f_1,$$

where Ω_1 and Ω_2 are the resonant frequencies of the network. These are given by

$$\Omega_1^2 = \frac{1}{a_2 - a_1}\left[\frac{(1 + a_2)}{L_1 C_o} - \frac{(1 + a_2)a_1}{L_1 C_2} - \frac{a_1}{L_2 C_2}\right] \tag{157}$$

$$= \frac{1}{2L_1C_2}\left(\frac{L_1}{L_2} + 1 + \frac{C_2}{C_o}\right) + \frac{1}{2L_1C_2}\left[\left(1+ \frac{L_1}{L_2} - \frac{C_2}{C_o}\right)^2 + 4\,\frac{C_2}{C_o}\right]^{1/2}$$

and

$$\Omega_2^2 = \frac{1}{2L_1C_2}\left(1+\frac{L_1}{L_2} + \frac{C_2}{C_o}\right) - \frac{1}{2L_1C_2}\left[\left(1+\frac{L_1}{L_2} - \frac{C_2}{C_o}\right)^2 + 4\,\frac{C_2}{C_o}\right]^{1/2}. \tag{158}$$

Since Ω_1 is greater than Ω_2, x_1 will represent the component of the solution at the driving frequency and x_2 the subharmonic response.

For convenience, let

$$\frac{1}{a_2 - a_1} = K_1\,, \quad \frac{1 + a_2}{a_2 - a_1} = K_2\,, \text{ and } \quad \frac{1 + a_1}{a_2 - a_1} = K_3\,. \tag{159}$$

The circuit differential equations can then be written as

$$\ddot{x}_1 + \Omega_1^2 x_1 = \epsilon K_2 f_1 - \epsilon K_1 f_2 = K_2\left[\frac{di}{dt} + \frac{R_1}{L_1}\,i\right. \tag{160}$$

$$\left. - \frac{(R + R_1)}{L_1}(\dot{x}_1 + \dot{x}_2) - \frac{D(x_1 + x_2)^2}{L_1 C_o}\right] - K_1\left[\frac{di}{dt}\right.$$

$$\left. + \frac{R_2}{L_2}\,i - \frac{R_2}{L_2}(\dot{x}_1 + \dot{x}_2) - \frac{R_2}{L_2}(a_1\dot{x}_1 + a_2\dot{x}_2)\right]$$

and

$$\ddot{x}_2 + \Omega_2^2 \ x_2 = \epsilon K_1 f_2 - \epsilon K_3 f_1 = K_1 \left[\frac{di}{dt} + \frac{R_2}{L_2} i \right. \tag{161}$$

$$\left. - \frac{R_2}{L_2} (\dot{x}_1 + \dot{x}_2) - \frac{R_2}{L_2} (a_1 \dot{x}_1 + a_2 \dot{x}_2) \right]$$

$$- K_3 \left[\frac{di}{dt} + \frac{R_1}{L_1} i - \frac{(R_2 + R_1)}{L} (\dot{x}_1 + \dot{x}_2) - \frac{D}{L_1 C_o} (x_1 + x_2)^2 \right] .$$

The terms on the right hand sides of (160) and (161), except $R_2 i$ and $R_1 i$, have all been assumed to be of the same small order. Hence, the above pair of quasilinear differential equations could be solved by perturbation series, equivalent linearization, or Kryloff's first approximation. The last method will be used. The terms

$$\frac{R_1}{L_1} i \text{ and } \frac{R_2}{L_2} \ i$$

are of order ϵ^2 and primarily lead to a small phase shift of the excitation. These terms are neglected in the following analysis.

The current is assumed to have the form $i = I \cos 2\omega t$. Then if the change of variable $\theta = \omega t$ is made, the pair of quasilinear equations become

$$x_1'' + 4 \ x_1 = \left(4 - \frac{\Omega_1^2}{\omega^2}\right) x_1 + \frac{K_2}{\omega^2} \left[- 2\omega I \sin 2\theta - \frac{(R + R_1)}{L_1} (x_1' + x_2') \omega \right. \tag{162}$$

$$\left. - \frac{D}{L_1 C_o} (x_1 + x_2)^2 \right] - \frac{K_1}{\omega^2} \left[- 2 I \sin 2\theta - \frac{R_2}{L_2} \omega (x_1' + x_2') \right.$$

$$\left. - \frac{R_2}{L_2} \omega (a_1 x_1' + a_2 x_2') \right] ,$$

$$x_2'' + x_2 = \left(1 - \frac{\Omega_2^2}{\omega^2}\right) x_2 + \frac{K_1}{\omega^2}\left[- 2\omega I \sin 2\theta - \frac{R_2}{L_2}\,\omega(x_1' + x_2')\right. \tag{163}$$

$$\left. - \frac{R_2}{L_2}\,\omega(a_1 x_1' + a_2 x_2')\right] - \frac{K_3}{\omega^2}\left[- 2\omega I \sin 2\theta - \frac{(R + R_1)}{L_1}\,\omega(x_1' + x_2')\right.$$

$$\left. - \frac{D}{L_1 C_o}\,(x_1 + x_2)^2\right] .$$

In Chapter II it was shown that equations of the above form possess solutions $x_1 = X_1 \sin(2\theta + \phi_1)$ and $x_2 = X_2 \sin(\theta + \phi_2)$ where ϕ_1, and ϕ_2 are determined by equations (43). For the system of equations above these are

$$\frac{d}{dt} X_1 = \frac{1}{4\pi\omega} \int_0^{2\pi} \left\{\left(4 - \frac{\Omega_1^2}{\omega^2}\right) x_1 + \frac{\epsilon K_2}{\omega^2} f_1(x_1, x_2, \theta)\right.$$

$$\left. - \frac{\epsilon K_1}{\omega^2} f_2(x_1, x_2, \theta)\right\} \cos(2\theta + \phi_1) d\theta$$

$$\frac{d\phi_1}{dt} = \frac{-1}{4\pi X_1 \omega} \int_0^{2\pi} \left\{\left(4 - \frac{\Omega_1^2}{\omega^2}\right) x_1 + \frac{\epsilon K_2}{\omega^2} f_1(x_1, x_2, \theta)\right.$$

$$\left. - \frac{\epsilon K_1}{\omega^2} f_2(x_1, x_2, \theta)\right\} \sin(2\theta + \phi_1) d\theta$$

$$\frac{dX_2}{dt} = \frac{1}{2\pi\omega} \int_0^{2\pi} \left\{\left(1 - \frac{\Omega_2^2}{\omega^2}\right) x_2 + \frac{\epsilon K_1}{\omega^2} f_2(x_1, x_2, \theta)\right.$$

$$\left. - \frac{\epsilon K_3}{\omega^2} f_1(x_1, x_2, \theta)\right\} \cos(\theta + \phi_2) d\theta ,$$

and

$$\frac{d\phi_2}{dt} = \frac{-1}{2\pi X_2 \omega} \int_0^{2\pi} \left\{\left(1 - \frac{\Omega_2^2}{\omega^2}\right) x_2 + \frac{\epsilon K_1}{\omega^2} f_2(x_1, x_2, \theta)\right.$$

$$\left. - \frac{\epsilon K_3}{\omega^2} f_1(x_1, x_2, \theta)\right\} \sin(\theta + \phi_2) d\theta .$$

It is understood that the zeroth approximations $x_1 = X_1 \sin(2\theta + \phi_1)$,

$x_2 = X_2 \sin(\theta + \phi_2)$ are substituted into the above integrals to evaluate them. If the indicated calculations are carried out, the equations of the first approximation which define X_1, X_2, ϕ_1 and ϕ_2, are

$$\frac{dX_1}{dt} = \frac{K_2}{2\omega^3}\left[\omega I \sin\phi_1 - \frac{2R\omega}{L_1} X_1 + \frac{DX_2^{\,2}}{4L_1C_o}\cos(2\phi_2 - \phi_1)\right] \tag{164}$$

$$- \frac{K_1}{2\omega^3}\left[\omega I \sin\phi_1 - \frac{2R_2}{L_1}\omega X_1 - \frac{R_2}{L_2}\omega a_1 X_1\right]$$

$$\frac{d\phi_1}{dt} = \frac{-1}{2\omega}\left(4 - \frac{\Omega_1^{\,2}}{\omega^2}\right) - \frac{K_2}{2\omega^3 X_1}\left[-\omega I \cos\phi_1 - \frac{DX_2^{\,2}}{4L_1C_o}\sin(2\phi_2 - \phi_1)\right] \tag{165}$$

$$- \frac{K_1 I}{2\omega^2 X_1}\cos\phi_1$$

$$\frac{dX_2}{dt} = \frac{K_1}{\omega^3}\left[-\frac{R_2}{L_2}\omega X_2 - \frac{R_2}{L_2}\omega a_2 X_2\right] - \frac{K_3}{\omega^3}\left[-\omega\frac{(R + R_1)}{L_1}X_2\right. \tag{166}$$

$$\left.- \frac{DX_1 X_2}{2L_1 C_o}\cos(2\phi_2 - \phi_1)\right]$$

$$\frac{d\phi_2}{dt} = -\frac{1}{\omega}\left(1 - \frac{\Omega_2^{\,2}}{\omega^2}\right) - \frac{K_3 D}{2\omega^3 L_1 C_o} X_1 \sin(2\phi_2 - \phi_1) \ . \tag{167}$$

In the above equations X_1, X_2, ϕ_1, and ϕ_2 are unknowns, which are to be determined from the equilibrium points of

$$\frac{dX_1}{dt} = \frac{dX_2}{dt} = \frac{d\phi_1}{dt} = \frac{d\phi_2}{dt} = 0 \ .$$

The equilibrium conditions are

$$(K_2 - K_1)\omega I \sin\phi_1 - \left[K_2 \frac{2R\omega}{L_1} - 2 K_1 \frac{R_2}{L_2}\omega(1 + a_1)\right] X_1 \tag{168}$$

$$+ \frac{K_2 D X_2^2}{4L_1 C_o} \cos(2\phi_2 - \phi_1) = 0$$

$$- \left(4 - \frac{\Omega_1^2}{\omega^2}\right)\omega^2 X_1 + (K_2 - K_1)\omega I \cos\phi_1 + \frac{K_2 D X_2^2}{4L_1 C_o} \sin(2\phi_2 - \phi_1) = 0 \tag{169}$$

$$- \left[K_1 \frac{R_2}{L_2}\omega(1 + a_2) - K_3 \frac{\omega}{L_1}(R + R_1)\right] \tag{170}$$

$$+ \frac{K_3 D X_2^2}{2L_1 C_o} \cos(2\phi_2 - \phi_1) = 0$$

$$- \left(1 - \frac{\Omega_2^2}{\omega^2}\right) - \frac{K_3 D X_1}{2\omega^2 L_1 C_o} \sin(2\phi_2 - \phi_1) = 0\ . \tag{171}$$

For X_1 and X_2 not equal to zero, these equations can be solved as follows. Equation (171) gives

$$\sin(2\phi_2 - \phi_1) = -\left(1 - \frac{\Omega_2^2}{\omega^2}\right)\frac{2\omega^2 L_1 C_o}{K_3 D X_1}\ .$$

If this is substituted into the third condition, X_1 is given by

$$\pm\left[\frac{K_3^2 D^2 X_1^2}{4L_1^2 C_o^2} - (\omega^2 - \Omega_2^2)^2\right]^{\frac{1}{2}} = K_1 \frac{R_2}{L_2}\omega(1 + a_2)$$

$$- K_3 \frac{\omega}{L_1}(R + R_1),$$

or

$$X_1 = \pm \frac{2L_1C_o}{K_3D}\left\{\left[K_1\frac{R_2}{L_2}(1 + a_2) - K_3\frac{\omega}{L_1}(R + R_1)\right]^2 + (\omega^2 - \Omega_2^{\ 2})^2\right\}^{\frac{1}{2}}. \tag{172}$$

Now if the identities for $\sin(2\phi_2 - \phi_1)$ and $\cos(2\phi_2 - \phi_1)$ found from (170) and (171) are substituted into (168) and (169), these are reducible to

$$X_1(K_2 - K_1)\omega I\sin\phi_1 = 2\left[K_2 R\frac{\omega}{L_1} - K_1\frac{R_2}{L_2}\omega(1 + a_1)\right]X_1^{\ 2} - \frac{K_2X_2^{\ 2}}{2K_3}\left[K_1\frac{R_2}{L_2}\omega(1 + a_2) - K_3\frac{\omega}{L_1}(R + R_1)\right] \tag{173}$$

and

$$X_1(K_2 - K_1)\omega I\cos\phi_1 = \omega^2X_1^{\ 2}\left(4 - \frac{\Omega_1^{\ 2}}{\omega^2}\right) - \frac{K_2X_2^{\ 2}}{2K_3}(\omega^2 - \Omega_2^{\ 2}) . \tag{174}$$

If equations (173), (174) are squared and added, a biquadratic equation for X_2 in terms of $X_1^{\ 2}$ results. If X_1 from (172) is substituted into this biquadratic, its solution $X_2^{\ 2}$ is

$$X_2^{\ 2} = \frac{8L_1C_o^{\ 2}}{K_2D^2K_3}\left\{(4\omega^2 - \Omega_1^{\ 2})(\omega^2 - \Omega_2^{\ 2}) + 2\omega^2\left[K_2\frac{R}{L_1} - K_1\frac{R_2}{L_2}(1 + a_1)\right]\left[K_1\frac{R_2}{L_2}(1 + a_2) - K_3\frac{(R + R_1)}{L_1}\right]\right\} \pm \frac{8L_1^{\ 2}C_o^{\ 2}\omega}{D^2K_3K_2}\left[\frac{K_3^{\ 2}D^2I^2}{4L_1C_o^{\ 2}}(K_2 - K_1)^2\right. \tag{175}$$

$$- \left\{ 2(\omega^2 - \Omega_2{}^2)\left[K_2 \frac{R}{L_1} - K_1 \frac{R_2}{L_2}(1 + a_1)\right] - (4\omega^2 - \Omega_1{}^2)\left[K_1 \frac{R_2}{L_2}(1 + a_2) - \frac{K_3}{L_1}(R + R_1)\right]\right\}^2 \Bigg]^{\frac{1}{2}} .$$

Unfortunately, the above result is so complex algebraically that it is very difficult to interpret physically. In order to obtain a readily interpretable result, coil resistances will be assumed small compared to R, the resistance of the nonlinear capacitor. This assumption is realistic, since the coils used possessed quality factors of about 200 while the quality factor of the nonlinear capacitor shunted by the tube was 20 to 40.

If terms in R_1 and R_2 are neglected, equation (172) gives

$$X_1{}^2 = \frac{4L_1{}^2 C_o{}^2}{K_3{}^2 D^2}\left[K_3^2 \frac{R^2\omega^2}{L_1{}^2} + (\omega^2 - \Omega_2{}^2)^2\right], \tag{176}$$

and equation (175) becomes

$$X_2{}^2 = \frac{8L_1{}^2C_o{}^2}{K_2 K_3 D^2}\left[(4\omega^2 - \Omega_1{}^2)(\omega^2 - \Omega_2{}^2) - 2\omega^2 K_2 K_3 \frac{R^2}{L_1{}^2}\right] \tag{177}$$

$$\pm \frac{8L_1{}^2C_o{}^2\omega}{D^2K_2K_3}\left\{\frac{K_3{}^2D^2I^2}{4L_1{}^2C_o{}^2}(K_2 - K_1)^2 - \left[2(\omega^2 - \Omega_2{}^2)K_2 \frac{R}{L_1} + (4\omega^2 - \Omega_1{}^2)\frac{K_3}{L_1}R\right]^2\right\}^{\frac{1}{2}}$$

where

$$K_1 = \frac{1}{a_2 - a_1}, \quad K_2 = \frac{1 + a_2}{a_2 - a_1}, \quad K_3 = \frac{1 + a_1}{a_2 - a_1},$$

and a_1, a_2 are given by equations (155), (156). The positive and negative signs denote two pairs of subharmonics which may exist. The positive sign of (177) denotes a pair of subharmonics differing 180° in phase, which exist and are stable provided the quantity within the braces is greater than zero. The negative sign denotes another possible pair of subharmonics which can exist only for proper detuning. This second pair of subharmonics exists, only if the bracketed terms are greater than ω times the radical.

For zero detuning

$$\omega = \frac{\Omega_1}{2} = \Omega_2 ,$$

the subharmonic response will exist if

$$DI > 4\omega C_o^2 R^2 \frac{K_2}{K_2 - K_1} = 4\omega C_o^2 R^2 \left(\frac{1}{a_2} + 1 \right) .$$

For any tuning condition no subharmonic exists unless

$$\frac{K_3^2 D^2 I^2 (K_2 - K_1)^2}{4L_1^2 C_o^2} \geq \left[(\omega^2 - \Omega_2^2) K_2 \frac{2R}{L_1} + (4\omega^2 - \Omega_1^2) \frac{K_3}{L_1} R \right]^2 . \tag{178}$$

If the above existence condition is met, subharmonics exist provided the right hand side of (177) is greater than zero.

If the network is properly adjusted so that Ω_1 is very near $2\Omega_2$, the subharmonic response is nearly a symmetrical function of frequency about $\omega = \Omega_2$.

The subharmonic and forced response solutions have been obtained, and some of their properties discussed. The first-order differential equations of the first approximation also have a solution $X_1 \neq 0$, $X_2 = 0$. This corresponds to the harmonic or forced solution without subharmonics. This harmonic solution and its properties will now be studied.

If terms in R_1 and R_2 are neglected, the first approximation equations reduce to

$$\frac{dX_1}{dt} = \frac{I}{2\omega^2}(K_2 - K_1)\sin\phi_1 - \frac{RK_2X_1}{\omega^2L_1} + \frac{K_2DX_2^2}{8\omega^3L_1C_o}\cos(2\phi_2 - \phi_1)$$

$$\frac{d\phi_1}{dt} = \frac{1}{2\omega}\left(4 - \frac{\Omega_1^2}{\omega^2}\right) + \frac{I}{2\omega^2X_1}(K_2 - K_1)\cos\phi_1 + \frac{K_2DX_2^2}{8\omega^3X_1L_1C_o}\sin(2\phi_2 - \phi_1)$$

$$\frac{dX_2}{dt} = \frac{K_3RX_2}{\omega^2L_1} + \frac{DK_3X_1X_2}{2\omega^3L_1C_o}\cos(2\phi_2 - \phi_1)$$

$$\frac{d\phi_2}{dt} = -\frac{1}{\omega}\left(1 - \frac{\Omega_2^2}{\omega^2}\right) - \frac{K_3DX_1}{2\omega^3L_1C_o}\sin(2\phi_2 - \phi_1)$$

Now if $X_2 = 0$, the solution X_1 is determined from the first two equations. So with $X_2 = 0$, the harmonic solution is determined from the equilibrium conditions

$$\frac{I}{2\omega^2}(K_2 - K_1)\sin\phi_1 - \frac{RK_2X_1}{2\omega^2L_1} = 0 \qquad (179)$$

and

$$-\frac{1}{2\omega}\left(4 - \frac{\Omega_1^2}{\omega^2}\right) + \frac{I}{2\omega^2X_1}(K_2 - K_1)\cos\phi_1 = 0\ . \qquad (180)$$

This pair of equations has the solution

$$X_1^2 = \frac{I^2(K_2 - K_1)^2 L_1^2}{4R^2K_2^2 + L_1^2\left(4\omega - \frac{\Omega_1^2}{\omega}\right)^2} \quad . \tag{181}$$

This is recognized as a linear circuit response. This response continues with $X_2 = 0$ until the driving current reaches a condition such that the harmonic solution with $X_2 = 0$ becomes unstable. Then X_1 no longer increases but X_2 builds up from rest. A detailed study of the stability of the solutions could be carried out from the first approximation equations by considering small perturbations of X_1, X_2, ϕ_1 and ϕ_2 from their equilibrium conditions.

The analysis of this section has shown a severe difficulty encountered with the perturbation and Kryloff approximation methods of analysis. This is the fact that even though accurate solutions can be found the algebraic complexity is often so great that the results cannot be interpreted. If a first approximation is sufficient the method of equivalent linearization can express the solution in terms of the effective linear resistance and reactance. Thus this method gives fewer details about the effects of the individual circuit elements but does yield a more compact result.

It is noted that in the above analysis the capacitor voltage was assumed to vary only as the first and second powers of charge. If the capacitor voltage had varied with the third power of charge to the same order as the second, the solution X_1 at 2ω would not have remained constant when $X_2 \neq 0$. This point is discussed further later in this chapter.

The circuit shown in Figure 26 was used to obtain experimental data

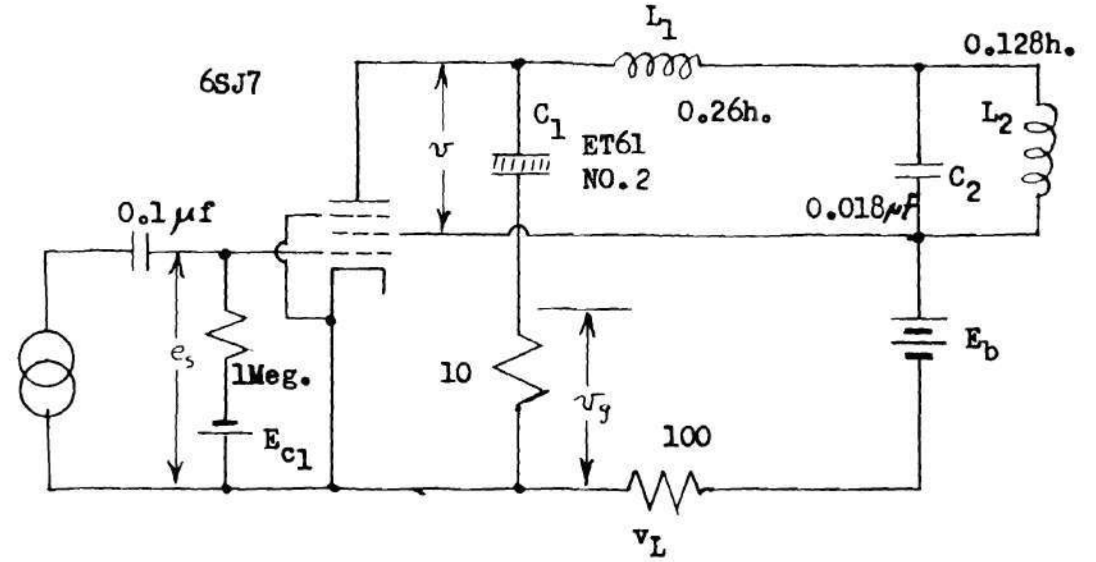


FIG. 26. EXPERIMENTAL SECOND-ORDER SUBHARMONIC CIRCUIT OF TWO DEGREES OF FREEDOM

on second-order subharmonic response with a resonant frequency near that of the excitation.

The ET61 dielectric was used in the nonlinear capacitor which had a capacity coefficient C_o of approximately 1900 mmf. The coils were decade inductors which were varied to maximize the plate voltage at 4 and 8 kilocycles. The excitation frequency was near 8 kilocycles, so that the network satisfied the tuning conditions for the second-order subharmonic. The capacitor current was determined by measuring the voltage across the 10-ohm series resistor with a harmonic wave analyzer. The current in the inductive branch was determined from the voltage developed across the 100 ohm resistor. The audio frequency plate voltage was metered with the harmonic analyzer using a very high resistance voltage divider across the plate network.

Measurements with the Freed "Q Meter" showed the plate circuit to have an effective Q of about 12 at 8 kc and 10 at 4 kc.

Amplitude response data were taken on this circuit by adjusting the input frequency so that $2\omega = \Omega_1 = 2\Omega_2$ as closely as possible, and by varying the audio grid voltage e_s in steps while the voltages v_q, v_p and v at both frequencies were read. Table 7 gives the data obtained for the circuit of Figure 26 with zero detuning. The 4 kc and 8 kc components of capacitor current are plotted against 8 kc grid voltage in Figure 27.

Figure 28 shows the variation of capacitor current with frequency at a constant grid voltage. The amount of frequency difference was determined by beating the input voltage e_s against a 4 kc reference generator and measuring the difference frequency by a Lissajous pattern with a third low-frequency oscillator. The frequency difference is

TABLE 7

Resonant Excitation of Second-Order Subharmonic Fig. 20 circuit, $Ec_1 = -5$, $E_b = 250$, 81°F, $\Delta F = 0$.

E_s volts at 8 kc	Ic in ma at 8 kc	Ic in ma at 4 kc	Il in ma at 8 kc	Il in ma at 4 kc	V volts at 8 kc	V volts at 4 kc
0	0	0	0	0	0	0
0.25	5.4	0	5.0	0	55.5	0
0.30	6.2	0.0	5.7	0.0	64.2	0.0
0.305	6.2	1.2	5.7	0.75	64.2	11.9
0.40	6.6	2.4	6.0	2.5	69.0	53.1
0.60	7.4	4.4	7.0	4.6	82.4	95.0
0.80	8.3	5.5	8.2	6.0	98.2	125
1.00	9.0	6.5	9.3	6.8	106	147
1.20	11.4	7.0	11.7	7.5	123	171
1.50	15.7	5.6	15.7	6.3	174	151
1.80	19.0	2.0	17.0	2.0	198	47.4
1.85	19.7	0.0	17.0	0.0	198	0

All Currents and Voltages are rms values.

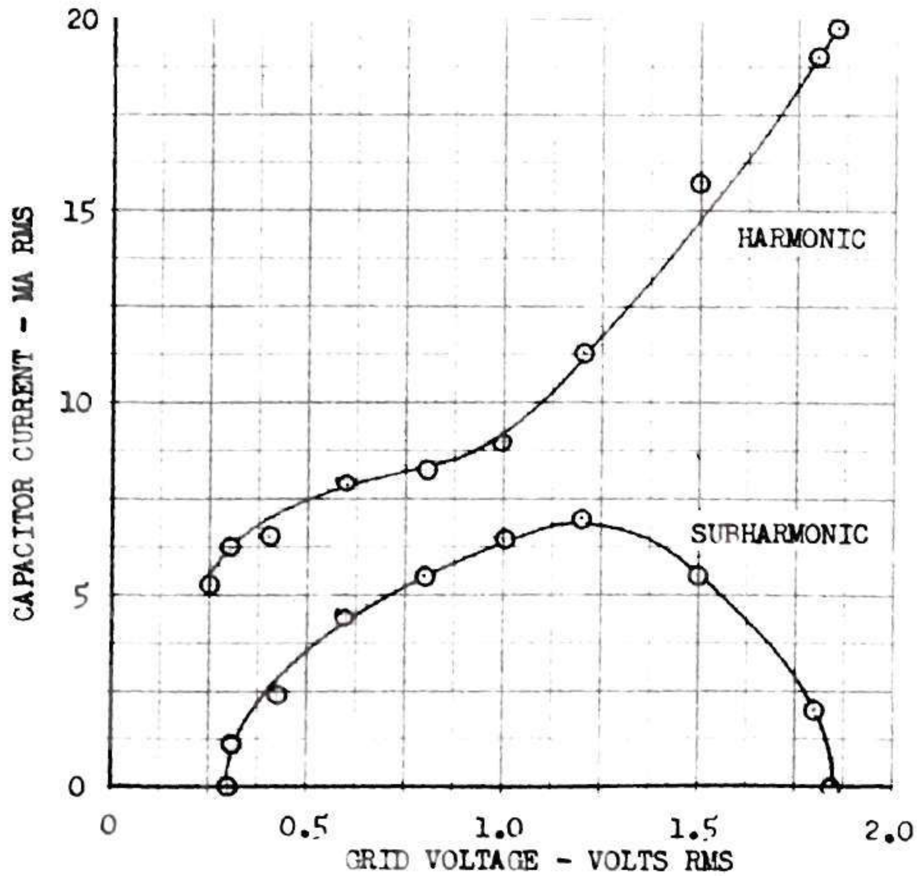


FIG. 27. HARMONIC AND SECOND SUBHARMONIC RESPONSE OF A DOUBLY RESONANT CIRCUIT

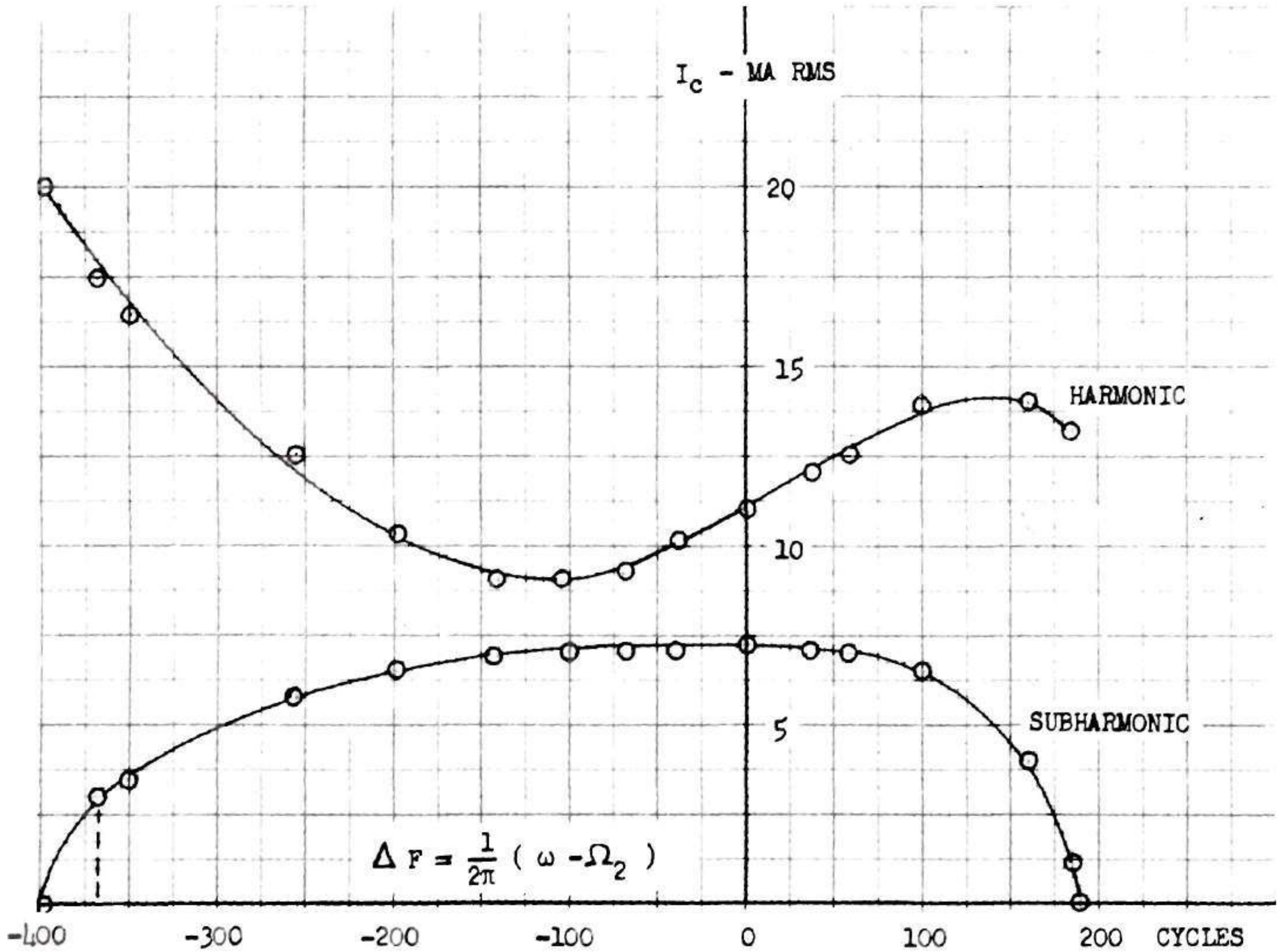


FIG. 28. VARIATION OF HARMONIC AND SUBHARMONIC CURRENTS
WITH FREQUENCY

$\Delta F = \frac{1}{2\pi}(\omega - \Omega_2)$. A slight hysteresis effect was observed at the lower frequency limit for the subharmonic. It was found that if C_2 was made much larger than the nonlinear capacity, the network could be tuned to Ω_1 and Ω_2 independently by varying L_1 and L_2, respectively. Table 8 gives data for two amplitude response runs at various fixed values of detuning.

If the frequency of e_s is changed to 16 kc, the tuning conditions are such as to allow the second-and fourth-order subharmonics to exist. It was found that the second-order subharmonic at 8 kc appears, if e_s is greater than 3.5 volts. The fourth-order subharmonic occurs also if e_s is greater than 4.4 volts.

The amplitude responses for $\Delta F = 0$, + 150 cycles, and - 248 cycles given by Figure 27 and Table 8 all show the subharmonic increases from zero to a maximum and then drops off to zero with continuously increasing input signal. Furthermore, the harmonic response increases with input grid voltage. However, Figure 27 shows a region of e_s in which the harmonic response varies slowly. Now, the tube current is approximately proportional to grid voltage; thus the experimental response characteristics do not agree with the analysis. The analytical study indicates that X_2, the subharmonic component of charge, should increase with input current while the harmonic component of charge X_1 should remain constant, when the subharmonic exists. It is probable that these differences in analytical and experimental results are due to two effects not considered in the analysis. First, dielectric heating could cause the subharmonic amplitude to decrease with increasing input current since the nonlinearity decreases with temperature. Second, and most important, finite cubic

Table 8. Second Subharmonic Response with Detuning.

$$\Delta F = \frac{1}{2\pi}(\omega - \Omega_2), \frac{\Omega_2}{2\pi} = 4 \times 10^3$$

$\Delta F = -248$ cps

e_s at 2ω	I_c ma at ω
0.405	0.0
0.412	1.2
0.60	3.9
0.80	5.5
1.00	6.2
1.20	6.1
1.50	5.5
1.65	4.6
1.70	2.5
2.00	2.4
2.50	2.1
3.00	0.0
	restarts
2.50	2.1

$\Delta F = 150$ cps

e_s at 2ω	I_c ma at ω
0.76	0
0.80	1.2
1.00	4.2
1.20	6.1
1.50	5.2
1.70	3.8
1.95	1.3
2.02	0

no hysteresis

curvature of the nonlinear characteristic could cause a reduction of subharmonic amplitude and a variation of the harmonic response.

In order to investigate the effects of cubic curvature of the nonlinear capacitance, the capacitor voltage-versus-charge characteristic is assumed to be

$$v = \frac{q}{C_o}\,(1 + D_1 q + D_2 q^2)\ .$$

If the coefficients D_1 and D_2 are both of order ϵ, the equations of the first approximation, with R_1 and R_2 terms neglected, are

$$\frac{dX_1}{dt} = \frac{I}{2\omega^2}\,(K_2 - K_1)\sin\phi_1 - \frac{RK_2}{\omega^2 L_1}\,X_1 + \frac{K_2 D_1 X_2^{\,2}}{8\omega^3 L_1 C_o}\cos(2\phi_2 - \phi_1), \tag{182}$$

$$\frac{d\phi_1}{dt} = -\frac{1}{2\omega}\,(4 - \frac{\Omega_1^{\,2}}{\omega^2}) + \frac{I(K_2 - K_1)}{2\omega^2 X_1}\cos\phi_1 \tag{183}$$

$$+ \frac{K_2 D_1 X_2^{\,2}}{8\omega^3 X_1 L_1 C_o}\sin(2\phi_2 - \phi_1) + \frac{3K_2 D_2}{8\omega^3 L_1 C_o}\,(\frac{X_1^{\,2}}{2} + X_2^{\,2})\ ,$$

$$\frac{dX_2}{dt} = \frac{K_3 R X_2}{\omega^2 L_1} + \frac{D_1 K_3 X_1 X_2}{2\omega^3 L_1 C_o}\cos(2\phi_2 - \phi_1)\ , \tag{184}$$

and

$$\frac{d\phi_2}{dt} = -\frac{1}{\omega}\,(1 - \frac{\Omega_2^{\,2}}{\omega^2}) - \frac{K_3 D_1 X_1}{2\omega^3 L_1 C_o}\sin(2\phi_2 - \phi_1) \tag{185}$$

$$- \frac{3K_3 D_2}{8\omega^3 L_1 C_o}\,(\frac{X_2^{\,2}}{2} + X_1^{\,2})\ .$$

The equilibrium conditions $\dot{X}_1 = \dot{\phi}_1 = \dot{X}_2 = \dot{\phi}_2 = 0$ define the first approximate solutions X_1 and X_2. Under these conditions (184) becomes

$$\cos(2\phi_2 - \phi_1) = - \frac{2\omega C_o R}{D_1 X_1} , \tag{186}$$

and substitution in (185) gives

$$- \frac{1}{\omega} (1 - \frac{\Omega_2^{\,2}}{\omega^2}) \mp K_3 \sqrt{\frac{D_1^{\,2} X_1^{\,2}}{4\omega^6 L_1^{\,2} C_o^{\,2}} - \frac{R^2}{\omega^4 L_1^{\,2}}} - \frac{3K_3 D_2}{8\omega^3 L_1 C_o} \left(\frac{X_2^{\,2}}{2} + X_1^{\,2} \right) = 0 ,$$

or

$$X_2^{\,2} = - 2X_1^{\,2} - \frac{16\omega^3 L_1 C_o}{3K_3 D_2} \left[\frac{1}{\omega} (1 - \frac{\Omega_2^{\,2}}{\omega^2}) \pm K_3 \sqrt{\frac{D_1^{\,2} X_1^{\,2}}{4\omega^6 L_1^{\,2} C_o^{\,2}} - \frac{R^2}{\omega^4 L_1^{\,2}}} \right] . \tag{187}$$

Now if the values of $\cos(2\phi_2 - \phi_1)$ and $\sin(2\phi_2 - \phi_1)$ as given by (184) and (185) are substituted, (182) and (183) become

$$0 = \frac{I(K_2 - K_1)}{2\omega^2} \sin \phi_1 - \frac{RK_2 X_1}{\omega^2 L_1} - \frac{K_2 X_2^{\,2} R}{4\omega^2 L_1 X_1} \tag{188}$$

and

$$0 = - \frac{1}{2\omega} (4 - \frac{\Omega_1^{\,2}}{\omega^2}) + \frac{I(K_2 - K_1)}{2\omega^2 X_1} \cos \phi_1 + \frac{3K_2 D_2}{8\omega^3 L_1 C_o} (\frac{X_1^{\,2}}{2} + X_2^{\,2}) \tag{189}$$

$$- \frac{K_2 X_2^{\;2}}{4K_3 X_1^{\;2}} \left[\frac{1}{\omega} \left(1 - \frac{\Omega_2^{\;2}}{\omega^2}\right) + \frac{3K_3 D_2}{8\omega^3 L_1 C_o} \left(\frac{X_2^{\;2}}{2} + X_1^{\;2} \right) \right] .$$

By equation (188),

$$\cos \phi_1 = \pm \left[1 - \frac{K_2^{\;2} R^2}{L^2 I^2 (K_2 - K_1)^2} \left(X_1 + \frac{X_2^{\;2}}{4X_1} \right)^2 \right]^{\frac{1}{2}} . \tag{190}$$

Substitution of (190) into (189) yields

$$0 = -\frac{1}{2\omega}\left(4 - \frac{\Omega_1^{\;2}}{\omega^2}\right) \pm \left\{ \frac{I^2 (K_2 - K_1)^2}{4\omega^4 X_1^{\;2}} - \frac{K_2^{\;2} R^2}{4\omega^4 L_1^{\;2} X_1^{\;4}} \left(X_1^{\;2} + \frac{X_2^{\;2}}{4} \right)^2 \right\}^{\frac{1}{2}}$$

$$+ \frac{3K_2 D_2}{8\omega^3 L_1 C_o} \left(\frac{X_1^{\;2}}{2} + X_2^{\;2} \right) - \frac{K_2 X_2^{\;2}}{4K_3 X_1^{\;2}} \left[\frac{1}{\omega} \left(1 - \frac{\Omega_2^{\;2}}{\omega^2}\right) + \frac{3K_3 D_2}{8\omega^3 L_1 C_o} \left(\frac{X_2^{\;2}}{2} + X_1^{\;2} \right) \right] \tag{191}$$

Equations (187) and (191) give the subharmonic and harmonic solutions for the charge on capacitor C_1 of Figure 27, since

$$q = x_1 + x_2 = X_1 \sin(2\theta + \phi_1) = X_2 \sin(\theta + \phi_2) .$$

Unfortunately, the algebraic complexity of equation (191) prevents explicit solutions from being found.

Equation (187) shows that the subharmonic solution, X_2, exists only if

$$\frac{D_1^{\;2} X_1^{\;2}}{4\omega^2 C_o^{\;2}} > R^2 ,$$

and

$$\frac{8\omega C_o}{3D_2}\sqrt{\frac{D_1^{\,2}X_1^{\,2}}{4\omega^2 C_o^{\,2}} - R^2} \geq X_1^{\,2} + \frac{8\omega^2 L_1 C_o}{3K_3 D_2}\left(1 - \frac{\Omega_2^{\,2}}{\omega^2}\right)^2 . \tag{192}$$

For zero detuning, (192) is reducible to

$$D_2^{\,2} X_1^{\,2} \geq \frac{8}{9} D_1^{\,2} - \sqrt{\frac{64}{81} D_1^{\,4} - \frac{64}{9} R^2 \omega^2 C_o^{\,2}} \tag{193}$$

and

$$D_2^{\,2} X_1^{\,2} \leq \frac{8}{9} D_1^{\,2} + \sqrt{\frac{64}{81} D_1^{\,4} - \frac{64}{9} R^2 \omega^2 C_o^{\,2}} . \tag{194}$$

These equations establish the upper and lower limits of X_1 for which $X_2 \neq 0$. These conditions define the limits of the region over which the subharmonic solution can exist. Thus it has been shown, even though a complete explicit solution could not be found, that a term of cubic curvature in the nonlinear capacitor characteristic could account for the observed upper limit on driving current at which the subharmonic exists.

The values of driving current I, at which X_1 is such that $X_2 = 0$ for $D_2 \neq 0$ can be found from the first approximation for $\dot{X}_1$ and $\dot{\phi}_1$. For $X_2 = 0$ and $D_2 \neq 0$, these are

$$\frac{dX_1}{dt} = \frac{I(K_2 - K_1)}{2\omega^2} \sin\phi_1 - \frac{RK_2}{\omega^2 L_1} X_1 \tag{195}$$

$$\frac{d\phi_1}{dt} = -\frac{1}{2\omega}\left(4 - \frac{\Omega_1^{\,2}}{\omega^2}\right) + \frac{I(K_2 - K_1)}{2\omega^2 X_1} \cos\phi_1 + \frac{3K_2 D_2 X_1^{\,2}}{16\omega^3 L_1 C_o} . \tag{196}$$

The equilibrium conditions $\dot{X}_1 = 0$, $\dot{\phi}_1 = 0$ yield

$$0 = -\frac{1}{2}\left(4 - \frac{\Omega_1^{\ 2}}{\omega^2}\right) \pm \left[\frac{I^2(K_2 - K_1)^2}{4\omega^4 X_1^{\ 2}} - \frac{R^2 K_2^{\ 2}}{\omega^4 L_1^{\ 2}}\right]^{1/2} + \frac{3K_2 D_2 X_2^{\ 2}}{16\omega^3 L_1 C_o} \quad .$$

Or for the case of zero detuning,

$$\frac{I^2(K_2 - K_1)^2}{4\omega^4} = \frac{R^2 K_2^{\ 2} X_1^{\ 2}}{\omega^4 L_1^{\ 2}} + \frac{9K_2^{\ 2} D_2^{\ 2} X_1^{\ 6}}{256\omega^6 L_1^{\ 2} C_o^{\ 2}} \quad . \tag{197}$$

Now if the limiting values of X_1 from (193) and (194) for which X_1 is just zero are substituted into the above equation, the critical values of I are obtained. Below the lower critical value or above the upper critical value the amplitude of the subharmonic component X_2 of capacitor charge is zero. Thus no subharmonic occurs if

$$I^2(K_2 - K_1)^2 \leq \frac{4R^2K^2}{D_2^{\ 2}L_1^{\ 2}}\left[\frac{8}{9}D_1^{\ 2} - \sqrt{\frac{64}{81}D_1^{\ 4} - \frac{64}{9}R^2\omega^2 C_o^{\ 2}}\right] + \frac{9K_2^{\ 2}}{64D_2^{\ 4}\omega^2 L_1^{\ 2} C_o^{\ 2}}\left[\frac{8}{9}D_1^{\ 2} - \sqrt{\frac{64}{81}D_1^{\ 4} - \frac{64}{9}R^2\omega^2 C_o^{\ 2}}\right]^3 \tag{198}$$

or

$$I^2(K_2 - K_1)^2 \geq \frac{4R^2K^2}{D_2^{\ 2}L_1^{\ 2}}\left[\frac{8}{9}D_1^{\ 2} + \sqrt{\frac{64}{81}D_1^{\ 4} - \frac{64}{9}R^2\omega^2 C_o^{\ 2}}\right] + \frac{9K_2^{\ 2}}{64D_2^{\ 4}\omega^2 L_1^{\ 2} C_o^{\ 2}}\left[\frac{8}{9}D_1^{\ 2} + \sqrt{\frac{64}{81}D_1^{\ 4} - \frac{64}{9}R^2\omega^2 C_o^{\ 2}}\right]^3 . \tag{199}$$

Furthermore the radicals must be real or

$$D_1^{\ 2} > 3R\omega C_o$$

if a subharmonic is to exist for any I.

The circuit of Figure 29 was used to obtain further experimental verification of the dependence of the subharmonic on the nonlinear characteristic. The bias potential across the nonlinear capacitor is $E_b - E_D$. Thus with the fixed tube-operating conditions $E_{c1} = -3$, $E_{c2} = 125$, and $E_b = 250$ volts the bias across the capacitor varies with E_D. Since the coefficients of the polynomial approximation to the capacitor characteristic vary with bias, the coefficients C_o, D_1 and D_2 are functions of E_D. The capacitor current and tube current were measured by reading v_p and v_g with the harmonic wave analyzer. For each run the value of E_D was fixed, and L_1 and L_2 were varied to maximize the tube plate voltage at 4 and 8 kc. The audio oscillator was set at 8 kc. For each value of E_D the circuit was retuned to make it resonant at the forcing frequency and its second order subharmonic. For each value of capacitor bias, the coefficients of the polynomial approximation to the capacitor characteristic were measured. The Freed Q meter was used to measure the capacitor Q at 4 and 8 kc for each value of E_D.

Table 9 gives the capacitor charge response for zero detuning and various values of E_D. In cases where hysteresis effects were observed, charges are given for increasing and decreasing values of e_s. The corresponding capacitor polynomial coefficients are given in Table 4 of Chapter III. Figures 30 and 31 give the subharmonic and harmonic capacitor currents for the alternating grid voltages and capacitor biases of Table 9. These curves show that, if e_s is sufficiently large, the subharmonic may disappear abruptly and the harmonic solution suddenly increase. A low-subharmonic-current region is observed to exist, for a capacitor bias of 100 volts, if e_s is increased above 3.85 volts and

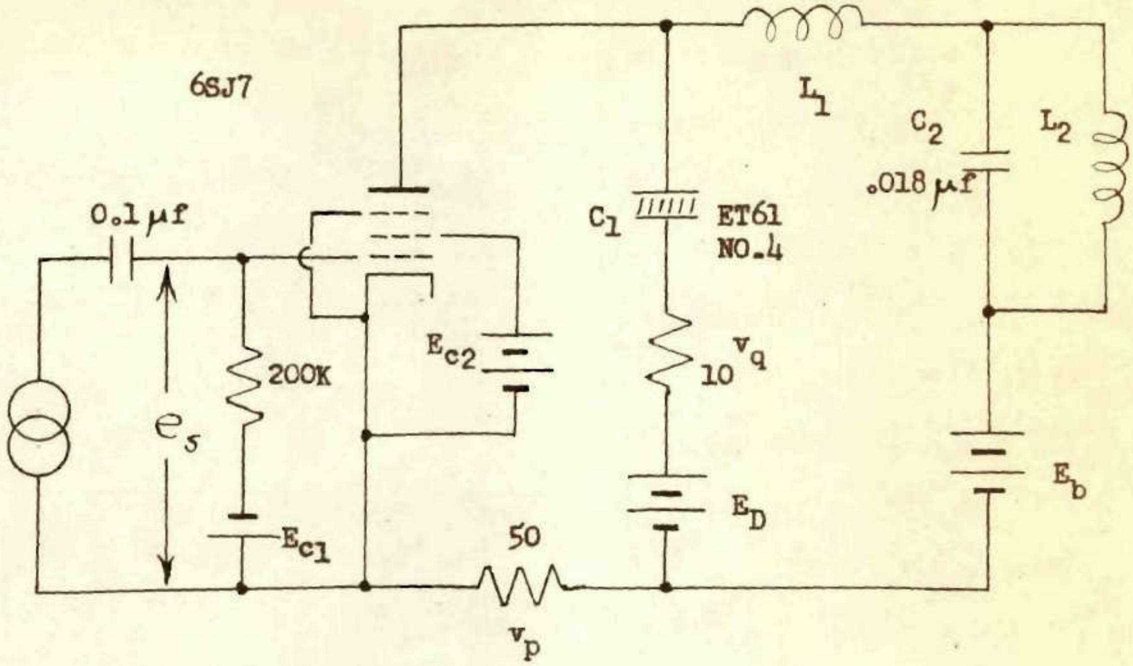


FIG. 29. EXPERIMENTAL CIRCUIT FOR OBSERVING THE SECOND-ORDER SUBHARMONIC WITH DIFFERENT NONLINER CHARACTERISTICS

Table 9. Subharmonic Response for Circuit of Figure 29

$$\omega = \Omega_1 = 2\Omega_2 = 16\pi \times 10^3,\ E_D = 250,\ C_2 = .018\mu f$$

$E_D = 0$				$E_D = 150$				$E_D = 200$			
$L_1 = 175$mh	$L_2 = 71$mh			$L_1 = 118$mh	$L_2 = 62$mh			$L_1 = 120$mh	$L_2 = 57$ mh		
e_s	I_c-ma		I_p-ma	e_s	I_c		I_p	e_s	I_c		I_p
at 2ω	at ω	at 2ω	at 2ω	at 2ω	at ω	at 2ω	at 2ω	at 2ω	at ω	at 2ω	at 2ω
0.1	0.0	4.7	0.15	0.09	0			0.3	0	6.2	0.46
0.2	1.85	5.7	0.30	0.30	1.8	4.4	0.38	0.4	2.4	7.4	0.68
0.3	3.2	6.7	0.53	0.4	3.6	5.2	0.73	0.5	3.5	8.0	0.88
0.4	4.1	7.6	0.77	0.8	6.0	6.3	1.50	0.6	4.3	8.7	1.06
0.6	5.6	8.9	1.20	1.2	7.5	6.8	2.20	0.7	5.0	9.8	1.28
0.8	6.8	10.0	1.76	1.6	9.0	7.3	2.83	0.8	5.2	11.8	1.46
1.0	7.7	11.3	2.1	2.0	10.4	8.4	3.50	0.88	0.0	17.5	1.58
1.2	8.6	12.1	2.5	2.4	11.0	10.7	3.94	0.74	5.1	9.9	1.32
1.4	9.3	13.3	2.8	2.8	10.6	13.2	4.24				
1.6	9.7	14.3	3.1	3.2	10.2	15.7	4.60				
1.8	9.3	18.0		3.6	9.4	17.5	4.80				
2.0	8.0	22.1		3.85	0	30.5	4.00				
2.2	7.7	23.6		2.55	1.1	29.5					
2.4	7.6	24.5		2.4	1.2	29.0					
				2.2	1.26	28.0					
				2.1	10.6	9.0					

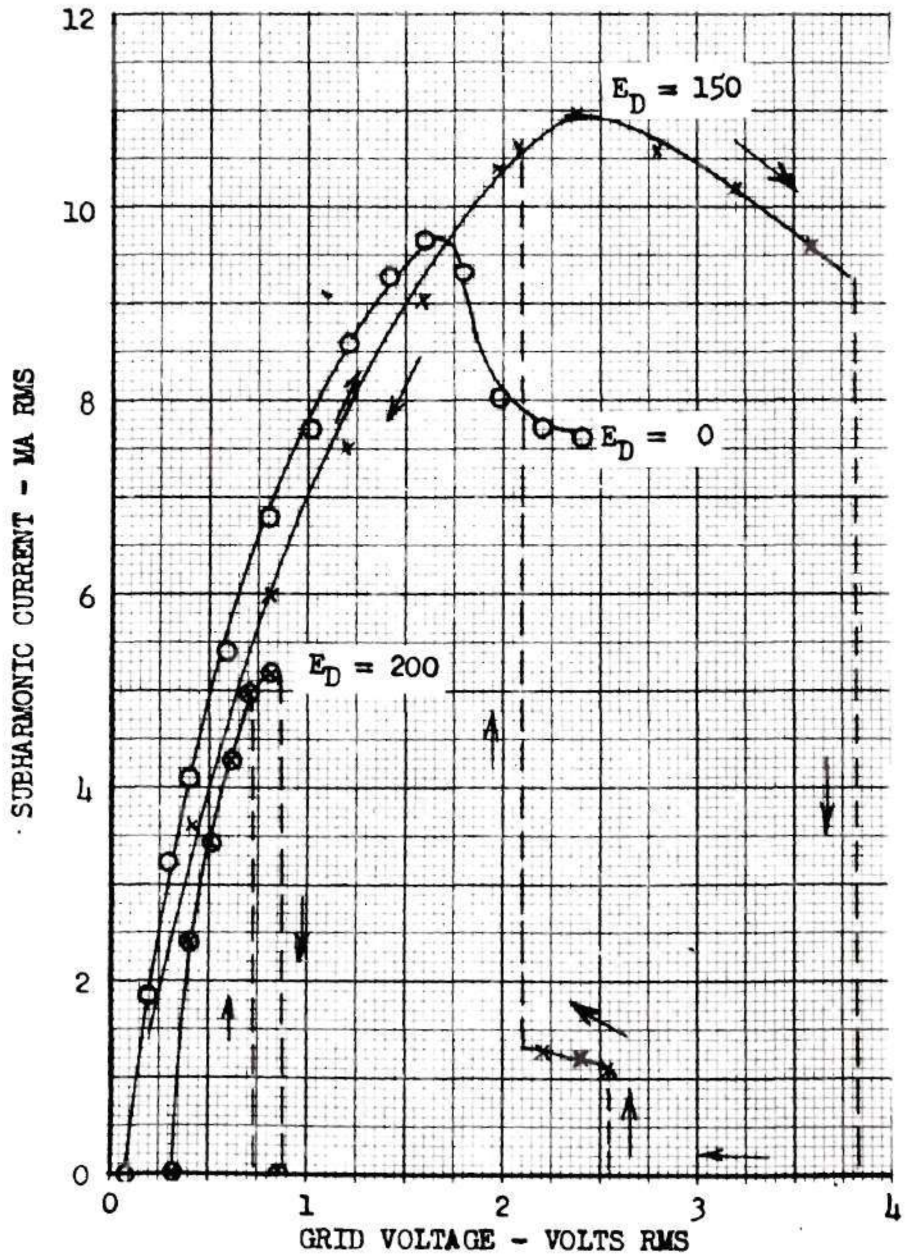


FIG. 30. SUBHARMONIC CAPACITOR CURRENT VERSUS GRID VOLTAGE

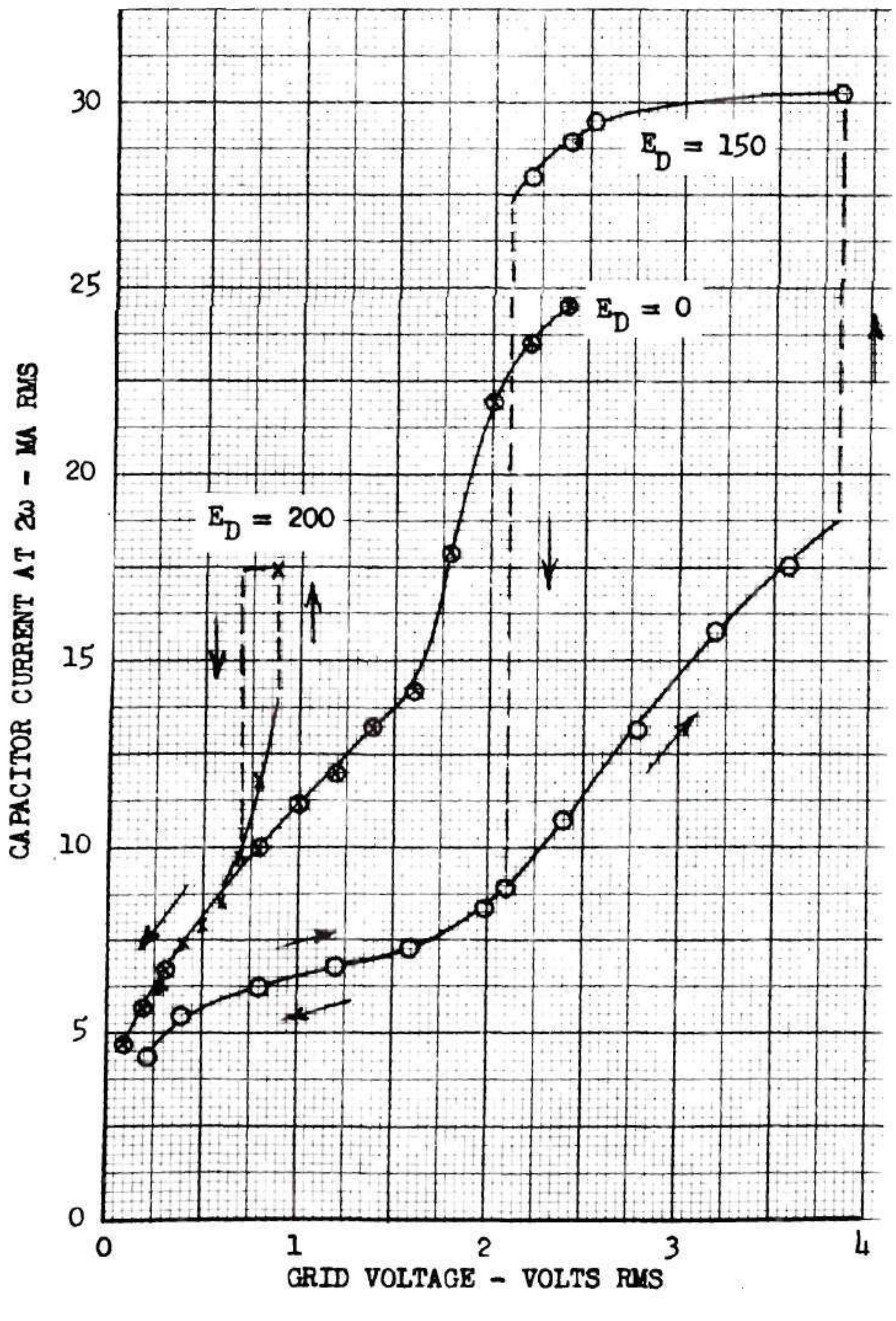


FIG. 31. HARMONIC CAPACITOR CURRENT VERSUS GRID VOLTAGE

then decreased. This low subharmonic current exists as e_s decreases from 2.4 volts to 2.1 volts. At the latter voltage the subharmonic returns to a high current condition and the harmonic abruptly decreases. The subharmonic versus harmonic capacitor current for each of the runs of Table 9 is plotted in Figure 32.

The response of the current-fed circuit of Figure 25 A has been considered in detail. The general properties of the dual voltage-fed circuit are quite similar. In fact, the analysis of this section could be applied to Figure 25 B by merely substituting $V/2\omega L_1$ for I. One practical property of the voltage-fed circuit is noteworthy. Since the circuit is resonant at the forcing frequency, the voltage required to excite the subharmonic is considerably reduced from that required in the single loop circuit. At audio frequencies the resistances of the nonlinear capacitors are of the same order of magnitude as the internal impedance of typical signal generators. Thus an audio oscillator used with a d.c. source for capacitor bias can be made to excite the second subharmonic in a voltage fed circuit, without the use of any vacuum tubes external to the audio oscillator.

A simple voltage-fed circuit is shown in Figure 33. A 5 kc subharmonic of a 10 kc input frequency was observed in this circuit if the voltage exceeded about 10 volts. Under these conditions the voltage across C_2 was almost a pure sine wave at 5 kc. This was the simplest circuit in which a subharmonic was obtained with the nonlinear capacitors. It is a very convenient circuit for the demonstration of second order subharmonics. However, the current-fed circuit is generally preferred for accurate experimental work since a greater range of amplitudes can be obtained and the internal impedance of the generator is not important in

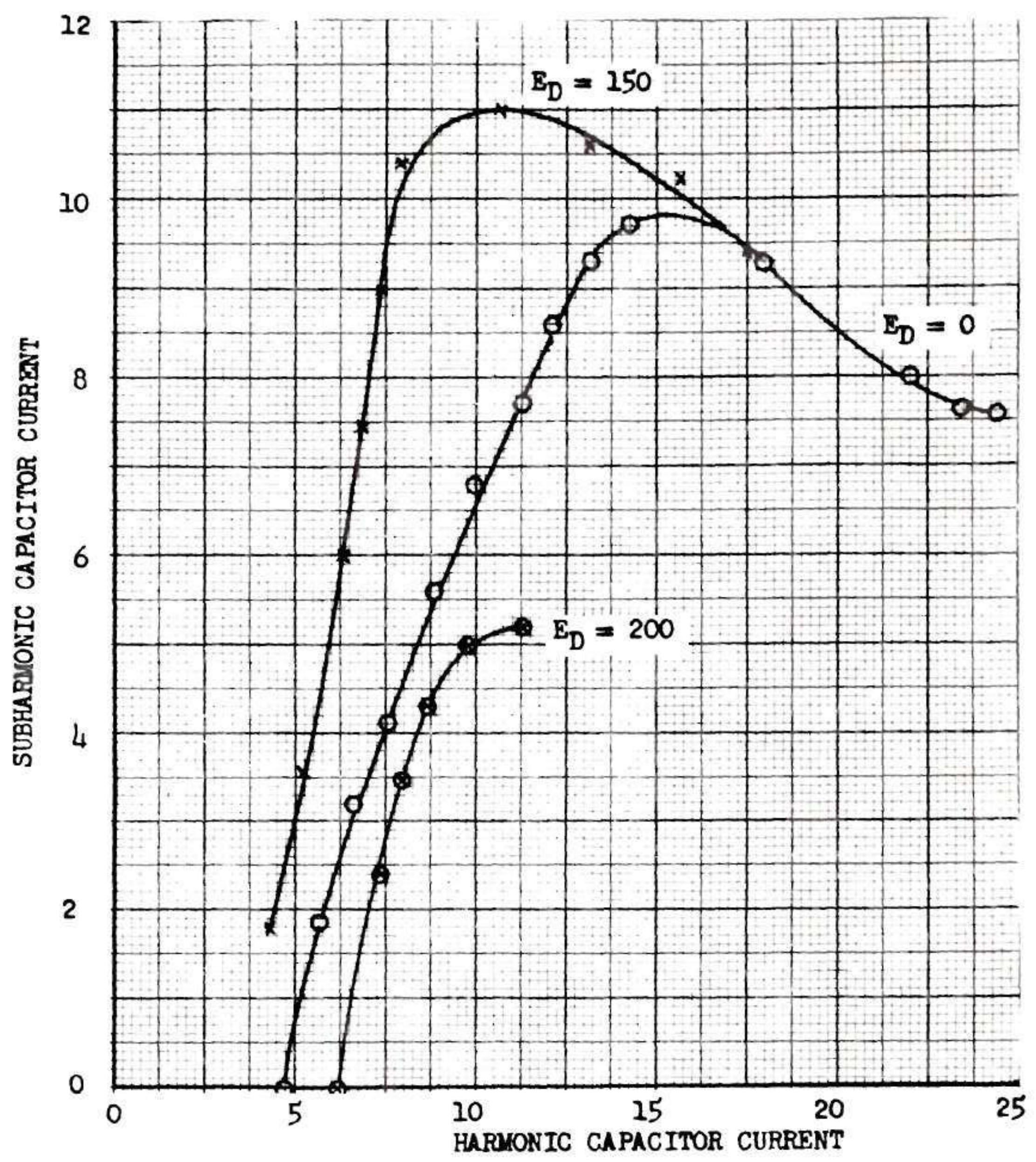


FIG. 32. SUBHARMONIC VERSUS HARMONIC CAPACITOR CURRENT

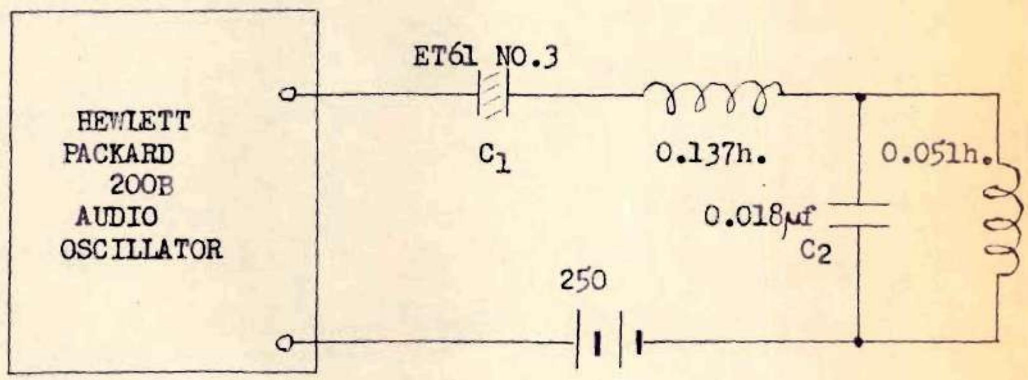


FIG. 33. A SIMPLE VOLTAGE-FED SECOND SUBHARMONIC CIRCUIT

analyzing the circuit response.

Irrational Solutions.

This section treats a class of response, observed by Manley[8] and Heegner[1], in nonlinear reactive circuits of two or more degrees of freedom, in which the principal frequencies involved are not rational fractions of the excitation frequency. This class of response is termed irrational. The irrational frequencies occur in pairs and their sum must be equal to the excitation frequency or a multiple thereof. The terms extract power from the driving source through cross modulation components. These "irrational" solutions are, of course, not subharmonics but are treated since their existence is important in the study of conditions under which subharmonics can be made to build up from rest.

A circuit capable of this class of response is shown in Figure 34. This circuit can have two resonant frequencies, as was shown in the previous section. It is assumed that these resonant frequencies are Ω_1 and Ω_2, neither of which is near the excitation frequency or a subharmonic thereof. It is further assumed that Ω_1 is not near a multiple of Ω_2. However, a degree of freedom at the excitation frequency could be added and still preserve this irrational response, provided that the resonances of the original two degrees of freedom are unchanged. The importance of the assumptions that neither Ω_1 nor Ω_2 is near $\frac{r}{s}\omega$, where r and s are integers less than or equal to the highest degree of nonlinearity, is treated in detail in the next section. It suffices here to assume that the tuning conditions are such that subharmonic resonance does not occur.

For convenience it is assumed that the resistances of the linear

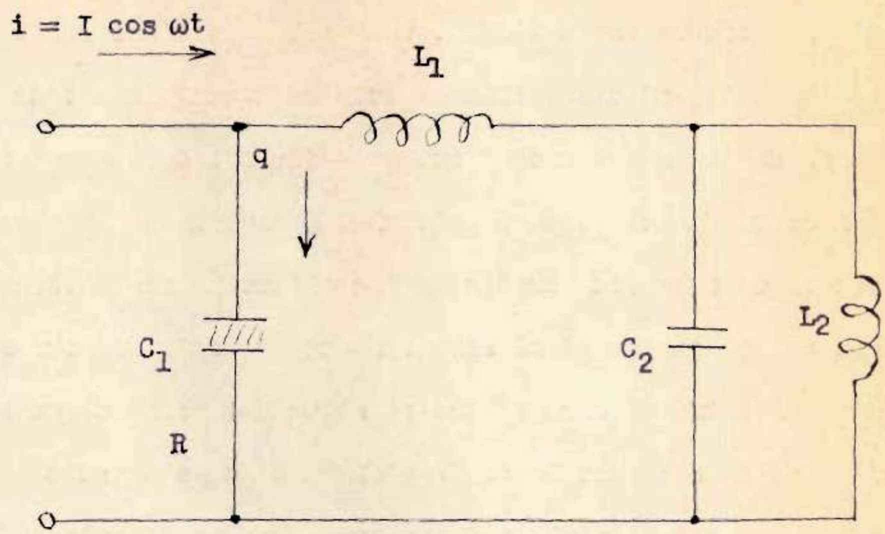


FIG. 34. DOUBLY RESONANT CURRENT FED CIRCUIT

elements L_1, L_2 and C_2 are negligible compared to the resistance, R, of the nonlinear capacitor C_1. If cubic curvature is neglected, the differential equations, whose solution is the charge on the capacitor C_1 of Figure 34, are

$$x_1'' + \frac{\Omega_1^2}{\omega^2} x_1 = -\frac{I}{\omega^2}(K_2 - K_1)\sin\theta - \frac{RK_2}{\omega L_1}(x_1' + x_2') - \frac{K_2 D}{\omega^2 L_1 C_o}(x_1 + x_2)^2 \tag{200}$$

$$x_2'' + \frac{\Omega_2^2}{\omega^2} x_2 = \frac{I}{\omega}(K_3 - K_1)\sin\theta + \frac{RK_3}{\omega^2 L_1}(x_1' + x_2') + \frac{DK_3}{\omega^2 L_1 C_o}(x_1 + x_2)^2 , \tag{201}$$

where

$$\Omega_1^2, \Omega_2^2 = \frac{1}{2L_1 C_2}\left(1 + \frac{L_1}{L_2} + \frac{C_2}{C_o}\right) \pm \left[\left(1 + \frac{L_1}{L_2} - \frac{C_2}{C_o}\right)^2 + 4\frac{C_2}{C_o}\right]^{\frac{1}{2}}.$$

It is now assumed that these equations possess solutions whose zeroth approximations for x_1 and x_2 are respectively

$$w_o = X_1 \sin(\omega_1 t + \phi_1) = X_1 \sin\left(\frac{\omega_1}{\omega}\theta + \phi_1\right) \tag{202}$$

and

$$y_o = X_2 \sin(\omega_2 t + \phi_2) = X_2 \sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right) . \tag{203}$$

Also, let $\omega = \omega_1 + \omega_2$. It is assumed that ω_1 and ω_2 are near Ω_1 and Ω_2, respectively. The responses possible for other relations between the frequency components will be considered later in this section.

Since neither Ω_1 nor Ω_2 is near ω, a second approximation solution will be sought. It is assumed that the nonlinearity and exciting current are of the order ϵ, a small parameter, and that $\Omega_1^2 - \omega_1^2$, $\Omega_2^2 - \omega_2^2$ and the losses are of order ϵ^2. The quasilinear differential equations (200) and (201) can then be rewritten as

$$x_1'' + \frac{\omega_1^2}{\omega^2} x_1 = \left(\frac{\omega_1^2}{\omega^2} - \frac{\Omega_1^2}{\omega^2} \right) x_1 - \frac{I}{\omega^2} (K_2 - K_1) \sin \theta \tag{204}$$

$$- \frac{RK_2}{\omega L_1} (x_1' + x_2') - \frac{K_2 D}{\omega^2 L_1 C_o} (x_1 + x_2)^2$$

$$= \epsilon H \sin \theta + \epsilon h_1(x_1, x_2) + \epsilon^2 h_2(x_1, x_1', x_2')$$

and

$$x_2'' + \frac{\omega_2^2}{\omega^2} x_2 = \left(\frac{\omega_2^2}{\omega^2} - \frac{\Omega_2^2}{\omega^2} \right) x_2 + \frac{I}{\omega^2} (K_3 - K_1) \sin \theta \tag{205}$$

$$+ \frac{RK_3}{\omega L_1} (x_1' + x_2') + \frac{DK_3}{\omega^2 L_1 C_o} (x_1 + x_2)^2$$

$$= \epsilon G \sin \theta + \epsilon g_1(x_1, x_2) + \epsilon^2 g_2(x_2, x_1', x_2')$$

respectively. These equations are of the form that can be treated by the second approximation for two degrees of freedom as developed in the Appendix. However, $\frac{r}{s}$ and $\frac{\ell}{k}$ are not rational fractions in this section. If

$$x_1 = w_o + \epsilon w_1 + \epsilon^2 w_2, \quad x_2 = y_o + \epsilon y_1 + \epsilon^2 y_2,$$

the equations which define the solutions to a first approximation are

$$\frac{\partial^2 w_1}{\partial \theta^2} + \frac{\omega_1^2}{\omega^2} w_1 = H \sin \theta + h_1(w_o, y_o) - 2 \frac{A_1}{\omega} \frac{\partial^2 w_o}{\partial\theta\,\partial X_1} \tag{206}$$

$$- 2 \frac{\sigma_1}{\omega} \frac{\partial^2 w_o}{\partial\theta\,\partial\phi_1} = - \frac{I}{\epsilon\varphi^2}(K_2 - K_1)\sin \theta - \frac{K_2 D}{\epsilon\omega^2 L_1 C_o}(w_o + y_o)^2$$

$$- 2 \frac{A_1}{\omega} \frac{\partial^2 w_o}{\partial\theta\,\partial X_1} - 2 \frac{\sigma_1}{\omega} \frac{\partial^2 w_o}{\partial\theta\,\partial\phi_1} = - \frac{I}{\epsilon\omega^2}(K_2 - K_1)\sin \theta$$

$$- \frac{K_2 D}{\epsilon\omega^2 L_1 C_o}\left[\frac{X_1^2}{2} - \frac{X_1^2}{2} \cos 2 \left(\frac{\omega_1}{\omega} \theta + \phi_1\right) + \frac{X_2^2}{2}\right.$$

$$- \frac{X_2^2}{2} \cos 2 \left(\frac{\omega_2}{\omega} \theta + \phi_2\right) + X_1 X_2 \cos\left(\frac{\omega_1 - \omega_2}{\omega} \theta + \phi_1 - \phi_2\right)$$

$$\left.- X_1 X_2 \cos(\theta + \phi_1 + \phi_2)\right] - 2 \frac{A_1}{\omega^2} \omega_1 \cos \left(\frac{\omega_1}{\omega} \theta + \phi_1\right)$$

$$+ 2 \frac{\sigma_1 X_1}{\omega^2} \omega_1 \sin\left(\frac{\omega_1}{\omega} \theta + \phi_1\right)$$

and

$$\frac{\partial^2 y_1}{\partial \theta^2} + \frac{\omega_2^2}{\omega^2} y_1 = G \sin \theta + g_1(w_o, y_o) - 2 \frac{B_1}{\omega} \frac{\partial^2 y_o}{\partial\theta\,\partial X_2} \tag{207}$$

$$- 2 \frac{\gamma_1}{\omega} \frac{\partial^2 y_o}{\partial\theta\,\partial\phi_2} = \frac{I}{\epsilon\omega^2} (K_3 - K_1)\sin \theta + \frac{K_3 D}{\omega^2 L_1 C_o} (w_o + y_o)^2$$

$$- 2 \frac{B_1}{\omega} \frac{\partial^2 y_o}{\partial\theta\,\partial X_2} - 2 \frac{\gamma_1}{\omega} \frac{\partial^2 y_o}{\partial\theta\,\partial\phi_2} = \frac{I(K_3 - K_1)}{\epsilon\omega^2} \sin \theta$$

$$- 2 \frac{B_1 \omega_2}{\omega^2} \cos\left(\frac{\omega_2}{\omega} \theta + \phi_2\right) + 2 \frac{\gamma_1 X_2 \omega_2}{\omega^2} \sin\left(\frac{\omega_2}{\omega} \theta + \phi_2\right)$$

$$+ \frac{K_3 D}{\epsilon\omega^2 L_1 C_o}\left[\frac{X_1^2}{2} - \frac{X_1^2}{2} \cos 2\left(\frac{\omega_1}{\omega} \theta + \phi_1\right) + \frac{X_2^2}{2}\right.$$

$$- \frac{X_2^2}{2} \cos 2\left(\frac{\omega_2}{\omega} \theta + \phi_2\right) + X_1 X_2 \cos\left(\frac{\omega_1 - \omega_2}{\omega}\theta + \phi_1 - \phi_2\right)$$

$$\left.- X_1 X_2 \cos(\theta + \phi_1 + \phi_2)\right] .$$

In order that w_1 and y_1 be periodic, it is necessary that terms of the frequencies ω_1 and ω_2 vanish in the w_1 and y_1 equations respectively. A sufficient condition is

$$A_1 = \sigma_1 = B_1 = \gamma_1 = 0 \quad . \tag{208}$$

Then

$$\begin{aligned} w_1 = &- \frac{I(K_2 - K_1)}{\epsilon(\omega_1^2 - \omega^2)} \sin\theta - \frac{K_2 D}{2\epsilon\omega^2 L_1 C_o} (X_1^2 + X_2^2) \\ &- \frac{K_2 D X_1^2}{6\epsilon L_1 C_o \omega_1^2} \cos 2\left(\frac{\omega_1}{\omega}\theta + \phi_1\right) - \frac{K_2 D}{\epsilon L_1 C_o} X_1 X_2 \\ &\left[\frac{\cos\left(\frac{\omega_1 - \omega_2}{\omega}\theta + \phi_1 - \phi_2\right)}{\omega_1^2 - (\omega_1 - \omega_2)^2} - \frac{\cos(\theta + \phi_1 + \phi_2)}{\omega_1^2 - \omega^2}\right] \\ &+ \frac{K_2 D X_2^2}{2\epsilon D_1 C_o} \frac{\cos 2\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)}{(\omega_1^2 - 4\omega_2^2)} , \end{aligned} \tag{209}$$

and

$$\begin{aligned} y_1 \quad & \frac{I(K_3 - K_1)}{\epsilon(\omega_2^2 - \omega^2)} \sin\theta + \frac{K_3 D}{2\omega^2 L_1 C_o \epsilon} (X_1^2 + X_2^2) + \frac{K_3 D}{\epsilon L_1 C_o} \\ &\left[- \frac{X_1^2 \cos 2\left(\frac{\omega_1}{\omega}\theta + \phi_1\right)}{2(\omega_2^2 - 4\omega_1^2)} + \frac{X_2^2}{6\omega_2^2} \cos 2\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)\right. \\ &\left. + \frac{X_1 X_2 \cos\left(\frac{\omega_1 - \omega_2}{\omega}\theta + \phi_1 - \phi_2\right)}{\omega_2^2 - (\omega_1 - \omega_2)^2} - \frac{X_1 X_2}{\omega_2^2 - \omega^2} \cos(\theta + \phi_1 + \phi_2)\right] . \end{aligned} \tag{210}$$

Since A_1, σ_1, B_1, γ_1 are zero and neither h_1 nor q_1 contains terms in x_1' or x_2', the equations which define the solutions for x_1 and x_2 to a second approximation are

$$\frac{\partial^2 w_2}{\partial \theta^2} + \frac{\omega_1^2}{\omega^2} w_2 = -2 \frac{A_2}{\omega} \frac{\partial^2 w_o}{\partial \theta \partial X_1} - 2 \frac{\sigma_2}{\omega} \frac{\partial^2 w_o}{\partial \theta \partial \phi_1} \tag{211}$$

$$+ h_2(w_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}) + \frac{\partial h_1}{\partial x_1}(w_o, y_o)\, w_1 + \frac{\partial h_1}{\partial x_2}(w_o, y_o) y_1 ,$$

and

$$\frac{\partial^2 y_2}{\partial \theta^2} + \frac{\omega_2^2}{\omega^2} y_2 = -2 \frac{B_2}{\omega} \frac{\partial^2 y_o}{\partial \theta \partial X_2} - 2 \frac{\gamma_2}{\omega} \frac{\partial^2 y_o}{\partial \theta \partial \phi_2} \tag{212}$$

$$+ g_2(y_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}) + \frac{\partial g_1}{\partial x_1}(w_o, y_o) w_1 + \frac{\partial g_1}{\partial x_2}(w_o, y_o) y_1 .$$

It is not necessary to calculate w_2 and y_2 to obtain solutions for x_1 and x_2 which satisfy the original quasi-linear pair of equations to a second approximation in ϵ. It is only necessary that w_2 and y_2 be periodic. Thus only the terms on the right hand sides of the w_2, y_2 equations, of angular frequencies ω_1 and ω_2, respectively, are needed. The sums of these terms must equal zero. Since

$$\frac{\partial h_1}{\partial X_1} = \frac{\partial h_1}{\partial X_2} = - \frac{2K_2 D}{\epsilon \omega^2 L_1 C_o} (w_o + y_o) ,$$

the terms from equation (211) of frequency ω_1 are

$$0 = -2 \frac{A_2 \omega_1}{2} \cos\left(\frac{\omega_1}{\omega} \theta + \phi_1\right) + 2 \frac{\sigma_2 \omega_1}{\omega^2} X_1 \sin\left(\frac{\omega_1}{\omega} \theta + \phi_1\right) \tag{213}$$

$$+ \frac{\omega_1^2 - \Omega_1^2}{\omega^2 \epsilon^2} X_1 \sin\left(\frac{\omega_1}{\omega} \theta + \phi_1\right) - \frac{RK_2}{\epsilon^2 \omega^2 L_1} X_1 \omega_1 \cos\left(\frac{\omega_1}{\omega} \theta + \phi_1\right)$$

$$- \frac{2K_2 D}{\epsilon^3 \omega^2 L_1 C_o} \left\{ \frac{(K_3 - K_2)}{2\omega^2 L_1 C_o} D X_1 (X_1^2 + X_2^2) \sin\left(\frac{\omega_1}{\omega} \theta + \phi_1\right) \right.$$

$$- \frac{DX_1^{\;3}}{2L_1C_o}\left[\frac{K_3}{6\omega_1^{\;2}} - \frac{K_2}{2(\omega_2^{\;2} - 4\omega_1^{\;2})}\right]\sin\left(\frac{\omega_1}{\omega}\theta + \phi_1\right)$$

$$+ \frac{IX_2}{2}\left[\frac{K_3 - K_1}{\omega_2^{\;2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\;2} - \omega^2}\right]\cos\left(\frac{\omega_1}{\omega}\theta - \phi_2\right)$$

$$+ \frac{X_1X_2^{\;2}D}{2L_1C_o}\left[\frac{K_3}{\omega_2^{\;2} - (\omega_2 - \omega_1)^2} - \frac{K_2}{\omega_1^{\;2} - (\omega_2 - \omega_1)^2}\right]\sin\left(\frac{\omega_1}{\omega}\theta + \phi_1\right)$$

$$- \frac{DX_1X_2^{\;2}}{2L_1C_o}\left[\frac{K_2}{\omega_1^{\;2} - \omega^2} - \frac{K_3}{\omega_2^{\;2} - \omega^2}\right]\sin\left(\frac{\omega_1}{\omega}\theta + \phi_1\right)\Bigg\} .$$

The corresponding terms of angular frequency ω_2 on the right hand side of equation (212) are

$$0 = -2\,\frac{B_2\omega_2}{\omega^2}\cos\left(\frac{\omega_2}{\omega}\theta + \phi_2\right) + 2\,\frac{\gamma_2\omega_2X_2}{\omega^2}\sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right) \qquad (214)$$

$$+ \frac{X_2}{\omega^2\epsilon^2}(\omega_2 - \Omega_2^{\;2})\sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right) + \frac{RK_3}{\epsilon^2\omega^2L_1}\omega_2X_2\cos\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)$$

$$+ \frac{2DK_3}{\epsilon^2\omega^2L_1C_o}\left\{\frac{(K_3 - K_2)}{2\omega^2L_1C_o}D\;X_2(X_1^{\;2} + X_2^{\;2})\sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)\right.$$

$$+ \frac{IX_1}{2}\left[\frac{K_3 - K_1}{\omega_2^{\;2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\;2} - \omega^2}\right]\cos\left(\frac{\omega_2}{\omega}\theta - \phi_1\right)$$

$$+ \frac{DX_1^{\;2}X_2}{2L_1C_o}\left[\frac{K_3}{\omega_2^{\;2} - (\omega_1 - \omega_2)^2} - \frac{K_2}{\omega_1^{\;2} - (\omega_1 - \omega_2)^2}\right]\sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)$$

$$+ \frac{DX_1^{\;2}X_2}{2L_1C_o}\left[\frac{K_3}{\omega_2^{\;2} - \omega^2} - \frac{K_2}{\omega_1^{\;2} - \omega^2}\right]\sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)$$

$$\left. + \frac{DX_2^{\;3}}{2L_1C_o}\left[\frac{K_3}{6\omega_2^{\;2}} - \frac{K_2}{2(\omega_1^{\;2} - 4\omega_2^{\;2})}\right]\sin\left(\frac{\omega_2}{\omega}\theta + \phi_2\right)\right\} .$$

By equation (213), it follows that

$$2\,\frac{A_2\omega_1}{\omega^2}\,\epsilon^2 = -\,\frac{RK_2X_1\omega_1}{\omega^2L_1} - \frac{K_2DX_2I}{\omega^2L_1C_o}\left[\frac{K_3 - K_1}{\omega_2^{\,2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\,2} - \omega^2}\right]\cos(\phi_1 + \phi_2), \tag{215}$$

and

$$\begin{aligned} -\,2\,\frac{\sigma_2\omega_1X_1\epsilon^2}{\omega^2} = {} & \frac{X_1}{\omega^2}\,(\omega_1^{\,2} - \Omega_1^{\,2}) - \frac{K_2D}{\omega^2L_1C_o}\left\{\frac{(K_3 - K_2)}{\omega^2L_1C_o}\,DX_1(X_1^{\,2} + X_2^{\,2})\right. \\ & - \frac{DX_1^{\,3}}{L_1C_o}\left[\frac{K_3}{6\omega_1^{\,2}} - \frac{K_2}{2(\omega_2^{\,2} - 4\omega_1^{\,2})}\right] + \frac{IX_2}{2}\left[\frac{K_3 - K_1}{\omega_2^{\,2} - \omega^2}\right. \\ & \left. - \frac{K_2 - K_1}{\omega_1^{\,2} - \omega^2}\right]\sin(\phi_1 + \phi_2) + \frac{X_1X_2^{\,2}D}{L_1C_o}\left[\frac{K_3}{\omega_2^{\,2} - (\omega_2 - \omega_1)^2}\right. \\ & \left.\left. - \frac{K_2}{\omega_1^{\,2} - (\omega_2 - \omega_1)^2}\right] - \frac{DX_1X_2^{\,2}}{L_1C_o}\left[\frac{K_2}{\omega_1^{\,2} - \omega^2} - \frac{K_3}{\omega_2^{\,2} - \omega^2}\right]\right\} . \end{aligned} \tag{216}$$

In a similar manner, equation (214) yields

$$2B_2\,\frac{\omega_2}{\omega^2}\,\epsilon^2 = \frac{RK_3X_2\omega_2}{\omega^2L_1} + \frac{K_3DX_1I}{\omega^2L_1C_o}\left[\frac{K_3 - K_1}{\omega_2^{\,2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\,2} - \omega^2}\right]\cos(\phi_1 + \phi_2), \tag{217}$$

and

$$-\,2\,\frac{\gamma_2\omega_2X_2\epsilon^2}{2} = \frac{X_2}{\omega^2}\,(\omega_2^{\,2} - \Omega_2^{\,2}) + \frac{K_3D}{\omega^2L_1C_o}\left\{\frac{(K_3 - K_2)}{\omega^2L_1C_o}\,DX_2(X_1^{\,2} + X_2^{\,2})\right. \tag{218}$$

$$+ \frac{DX_2^{\,2}}{L_1C_o}\left[\frac{K_3}{6\omega_2^{\,2}} - \frac{K_2}{2(\omega_1^{\,2} - 4\omega_2^{\,2})}\right] + IX_1\left[\frac{K_3 - K_1}{\omega_2^{\,2} - \omega^2}\right.$$

$$\left. - \frac{K_2 - K_1}{\omega_1^{\,2} - \omega^2}\right]\sin(\phi_1 + \phi_2) + \frac{DX_1^{\,2}X_2}{L_1C_o}\left[\frac{K_3}{\omega_2^{\,2} - (\omega_1 - \omega_2)^2}\right.$$

$$\left. - \frac{K_2}{\omega_1^{\,2} - (\omega_1 - \omega_2)^2}\right] + \frac{DX_1^{\,2}X_2}{L_1C_o}\left[\frac{K_3}{\omega_2^{\,2} - \omega^2} - \frac{K_2}{\omega_1^{\,2} - \omega^2}\right]\Bigg\} \quad .$$

The values of X_1, X_2, ϕ_1 and ϕ_2, for which $w_o + \epsilon w_1$ and $y_o + \epsilon y_1$ satisfy the original differential equations to an accuracy of ϵ^2, are defined by the equilibrium points of four first-order differential equations. These are

$$\frac{da_1}{dt} = \epsilon A_1 + \epsilon^2 A_2 = 0 \quad , \quad \frac{d\phi_1}{dt} = \epsilon\,\sigma_1 + \epsilon^2\sigma_2 = 0 \; ,$$

$$\frac{da_2}{dt} = \epsilon\, B_1 + \epsilon^2 B_2 = 0 \; , \quad \frac{d\phi_2}{dt} = \epsilon\,\gamma_1 + \epsilon^2\,\gamma_2 = 0 \quad .$$

Since

$$A_1 = \sigma_1 = B_1 = \gamma_1 = 0 \quad ,$$

the values of X_1, X_2, ϕ_1, ϕ_2 can be determined by simultaneous solution of the four equations obtained by equating A_2, σ_2, B_2 and γ_2 to zero. The solutions $w_o + \epsilon w_1$, and $y_o + \epsilon y_1$ exist if, for the values of X_1, X_2, ϕ_1, ϕ_2 which satisfy $A_2 = \sigma_2 = B_2 = \gamma_2 = 0$, the Jacobian

$$J\left(\frac{A_2, \sigma_2, B_2, \gamma_2}{X_1, X_2, \phi_1, \phi_2}\right) \neq 0 \quad ,$$

that is if the four equations are independent. The nonvanishing condition on the Jacobian furnishes an existence condition for the solutions. This point is discussed in more detail in Chapters II and III.

The four equations to be solved for X_1, ϕ_1, X_2, ϕ_2 are

$$0 = RX_1\omega_1 + \frac{DX_2I}{C_o}\left[\frac{K_3 - K_1}{\omega_2^2 - \omega^2} - \frac{K_2 - K_1}{\omega_1^2 - \omega^2}\right]\cos(\phi_1 + \phi_2) \tag{219}$$

$$0 = RX_2\omega_2 + \frac{DX_1I}{C_o}\left[\frac{K_3 - K_1}{\omega_2^2 - \omega^2} - \frac{K_2 - K_1}{\omega_1^2 - \omega^2}\right]\cos(\phi_1 + \phi_2) \tag{220}$$

$$\begin{aligned} 0 = X_1(\omega_1^2 - \Omega_1^2) - \frac{K_2D}{L_1^2C_o^2}\Bigg\{ & \frac{(K_3 - K_2)}{\omega^2} DX_1(X_1^2 + X_2^2) \\ & - DX_1^3\left[\frac{K_3}{6\omega_1^2} - \frac{K_2}{2(\omega_2^2 - 4\omega_1^2)}\right] + IX_2L_1C_o\left[\frac{K_3 - K_1}{\omega_2^2 - \omega^2}\right. \\ & \left. - \frac{K_2 - K_1}{\omega_1^2 - \omega^2}\right]\sin(\phi_1 + \phi_2) + DX_1X_2^2\left[\frac{K_3}{\omega_2^2 - (\omega_2 - \omega_1)^2}\right. \\ & \left. + \frac{K_3}{\omega_2^2 - \omega^2} - \frac{K_2}{\omega_1^2 - (\omega_2 - \omega_1)^2} - \frac{K_2}{\omega_1^2 - \omega^2}\right]\Bigg\} . \end{aligned} \tag{221}$$

$$\begin{aligned} 0 = X_2(\omega_2^2 - \Omega_2^2) + \frac{K_3D}{L_1^2C_o^2}\Bigg\{ & \frac{K_3 - K_2}{\omega^2} DX_2(X_1^2 + X_2^2) \\ & - DX_2^3\left[\frac{K_3}{6\omega_2^2} - \frac{K_2}{2(\omega_1^2 - 4\omega_2^2)}\right] + IX_1L_1C_o\left[\frac{K_3 - K_1}{\omega_2^2 - \omega^2}\right. \\ & \left. - \frac{K_2 - K_1}{\omega_1^2 - \omega^2}\right]\sin(\phi_1 + \phi_2) \end{aligned} \tag{222}$$

$$+ DX_1^2 X_2 \left[\frac{K_3}{\omega_2^2 - (\omega_2 - \omega_1)^2} + \frac{K_3}{\omega_2^2 - \omega^2} - \frac{K_2}{\omega_1^2 - (\omega_2 - \omega_1)^2} - \frac{K_2}{\omega_1^2 - \omega^2} \right] \Bigg\} .$$

These four equations have as one solution $X_1 = X_2 = 0$. This corresponds to the harmonic response at the frequency ω. This solution is of interest in studying the threshold condition above which X_1 and X_2 are not zero and will be treated later in this section. For the present it is assumed that neither X_1 nor X_2 is zero, and another solution is sought for the four equations.

If neither X_1 nor X_2 equals zero, equation (219) divided by X_2 less equation (220) divided by X_1 yields

$$0 = \frac{RX_1}{X_2} \omega_1 - \frac{RX_2}{X_1} \omega_2 \quad ,$$

or

$$\frac{X_1^2}{X_2^2} = \frac{\omega_2}{\omega_1} \quad . \tag{223}$$

Hence, combining equations (219) and (223) yields

$$\cos(\phi_1 + \phi_2) = - \frac{R \sqrt{(\omega_1 \omega_2)} \quad C_o}{DI \left\{ \left[\frac{K_3 - K_1}{\omega_2^2 - \omega^2} \quad \frac{K_2 - K_1}{\omega_1^2 - \omega^2} \right] \right.} \quad . \tag{224}$$

When (223) and (224) are substituted into (221), it reduces to

$$0 = X_1(\omega_1{}^2 - \Omega_1^2) - \frac{K_2 D}{L_1{}^2 C_o{}^2}\Bigg\{\frac{(K_3 - K_2)}{\omega^2} DX_1{}^3(1 + \frac{\omega_1}{\omega_2}) \tag{225}$$

$$- DX_1{}^3\left[\frac{K_3}{6\omega_1{}^2} - \frac{K_2}{2(\omega_2{}^2 - 4\omega_1{}^2)}\right] \pm \sqrt{\frac{\omega_1}{\omega_2}}\, X_1 L_1 C_o \left[I^2\left(\frac{K_3 - K_1}{\omega_2{}^2 - \omega^2}\right.\right.$$

$$\left.\left. - \frac{K_2 - K_1}{\omega_1{}^2 - \omega^2}\right)^2 - \frac{R^2\omega_1\omega_2 C_o{}^2}{D^2}\right]^{\frac{1}{2}} + D\frac{\omega_1}{\omega_2} X_1{}^3\left[\frac{K_3}{\omega_2^2 - (\omega_2 - \omega_1)^2}\right.$$

$$\left. + \frac{K_3}{\omega_2{}^2 - \omega^2} - \frac{K_2}{\omega_1{}^2 - (\omega_2 - \omega_1)^2} - \frac{K_2}{\omega_1{}^2 - \omega^2}\right]\Bigg\} \quad .$$

Thus

$$X_1{}^2\Bigg\{\frac{-K_3 + K_2}{\omega^2} D(1 + \frac{\omega_1}{\omega_2}) - D\left[\frac{-K_3}{6\omega_1{}^2} + \frac{K_2}{2(\omega_2{}^2 - 4\omega_1{}^2)}\right] \tag{226}$$

$$+ D\frac{\omega_1}{\omega_2}\left[\frac{-K_3}{\omega_2{}^2 - (\omega_2 - \omega_1)^2} - \frac{K_3}{\omega_2{}^2 - \omega^2} + \frac{K_2}{\omega_1{}^2 - (\omega_2 - \omega_1)^2}\right.$$

$$\left. + \frac{K_2}{\omega_1{}^2 - \omega^2}\right]\Bigg\}$$

$$= - \frac{L_1{}^2 C_o{}^2}{K_2 D}(\omega_1{}^2 - \Omega_1{}^2)$$

$$\pm \sqrt{\frac{\omega_1}{\omega_2}}\, L_1 C_o\left[I^2\left(\frac{K_3 - K_1}{\omega_2{}^2 - \omega^2} - \frac{K_2 - K_1}{\omega_1{}^2 - \omega^2}\right)^2 - \frac{R^2\omega_1\omega_2 C_o{}^2}{D^2}\right]^{\frac{1}{2}},$$

where

$$K_1 = \frac{1}{a_2 - a_1}\,,\ K_2 = \frac{a_2 + 1}{a_2 - a_1}\,,\ K_3 = \frac{a_1 + 1}{a_2 - a_1}\ ,$$

and

$$a_1, a_2 = -\frac{1}{2}\left(1 + \frac{L_1}{L_2} - \frac{C_2}{C_o}\right) \mp \frac{1}{2}\left[\left(1 + \frac{L_1}{L_2} - \frac{C_2}{C_o}\right)^2 + 4\ \frac{C_2}{C_o}\right]^{1|2}.$$

Since ϕ_1 and ϕ_2 occur only as $\phi_1 + \phi_2$, the equations (219), (220) (221) are sufficient to determine the unknowns X_1, X_2, $\phi_1 + \phi_2$ as shown above. Equation (222) has not been used in finding the solution but it must still be satisfied by the X_1, X_2, $\phi_1 + \phi_2$ values given by (223), (224), (226). It is possible to satisfy (222), since neither ω_1 nor ω_2 has been specified but only their sum, as $\omega_1 + \omega_2 = \omega$. If three of the conditions (219), (220), (221), (222) are used to find X_1, X_2, $\phi_1 + \phi_2$ in terms of ω, ω_1, ω_2 and circuit constants, the fourth determines the frequency ω_1 or ω_2 of the solution. This is a situation similar to the free oscillation of a vacuum tube oscillator, in which the frequency is fixed by a balance of reactive power rather than by the driving frequency as in subharmonic resonance. In fact, the responses at pairs of irrational frequencies could be considered a class of free oscillation occurring in driven passive circuits with nonlinear reactive elements.

Equation (221) divided by X_2 less equation (222) divided by X_1 gives

$$0 = \frac{-X_1}{K_2X_2}(\omega_1 - \Omega_1^2) + \frac{X_2}{K_3X_1}(\omega_2^2 - \Omega_2^2) \tag{227}$$

$$+ \frac{D^2}{\omega^2 L_1^2 C_o^2}(K_3 - K_2)\left(\frac{X_2}{X_1} - \frac{X_1}{X_2}\right)(X_1^2 + X_2^2)$$

$$- \frac{D^2}{L_1^2 C_o^2}\left(\frac{X_2^3}{X_1} - \frac{X_1^3}{X_2}\right)\left[\frac{K_3}{6\omega_2^2} - \frac{K_2}{2(\omega_1^2 - 4\omega_2^2)}\right].$$

With

$$X_2 \quad \frac{\omega_1}{\omega_2} X_1 \quad ,$$

this reduces to

$$0 = \frac{-1}{K_2}\sqrt{\frac{\omega_2}{\omega_1}} \; (\omega_1{}^2 - \Omega_1{}^2) + \frac{1}{K_3}\sqrt{\frac{\omega_1}{\omega_2}} \; (\omega_2{}^2 - \Omega_2{}^2) \tag{228}$$

$$+ \frac{D^2 X_1{}^2}{L_1{}^2 C_o{}^2}\left\{\frac{K_3 - K_2}{\omega^2}\left[\sqrt{\frac{\omega_1}{\omega_2}} - \sqrt{\frac{\omega_2}{\omega_1}}\right](1 + \frac{\omega_1}{\omega_2})\right.$$

$$\left. - \left[\left(\frac{\omega_1}{\omega_2}\right)^{3|2} - \left(\frac{\omega_2}{\omega_1}\right)^{1|2}\right]\left[\frac{K_3}{6\omega_2^2} - \frac{K_2}{2(\omega_1{}^2 - 4\omega_1{}^2)}\right]\right\} .$$

This equation, in combination with the equation for X_1 in terms of I, fixes the value of ω_1 and hence $\omega_2 = \omega - \omega_1$. This is not carried out in detail, since its algebraic complexity prevents a general solution. It is noted that as X_1 and X_2 approach zero the frequency ω_1 is a solution of

$$0 = \frac{-1}{K_2} \; \frac{\omega - \omega_1}{\omega_1}(\omega_1{}^2 - \Omega_1{}^2) + \frac{1}{K_3}\left[(\omega - \omega_1{}^2) - \Omega_2{}^2\right].$$

Thus values of X_1, X_2, ϕ_1 and ϕ_2, which satisfy the equilibrium conditions (219), (220), (221), (222), have been found. Since ϕ_1 and ϕ_2 occur in these solutions only as their sum, small perturbations of ϕ_1 or ϕ_2 about their equilibrium values have the same effect. So the roots of the characteristic equation, formed by equating to zero the determinant of the coefficients of small perturbations about the equilibrium points, have zero real parts. Thus the stability of these solutions cannot be determined by considering only first powers of the small perturbations from equilibrium. Liapounoff[52] has shown in such cases that higher order terms of the perturbations must be included to determine the stability. However, a physical interpretation of the problem can be given from observations of experimental responses. The failure of the first approximation stability analysis apparently indicates that the phases will not return to their unperturbed equilibrium conditions and that the amplitudes will not decay to zero with time. That is, the phases of ϕ_1 and ϕ_2 are not independently fixed and if the circuit conditions are changed the frequencies ω_1 and ω_2 will change also.

From equation (226), since $K_3 < 0$ and the coefficient of $X_1^{\,2}$ is normally positive, the conditions that the solution $X_1 \neq 0$, $X_2 \neq 0$ exist are

$$I\left|\frac{K_3 - K_1}{\omega_2^{\,2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\,2} - \omega^2}\right| D > R\,\omega_1\omega_2\,C_o. \tag{229}$$

and

$$0 < -\frac{L_1^{\,2}C_o^{\,2}}{K_2 D}\left(\omega_1^{\,2} - \Omega_2^{\,2}\right) \pm \sqrt{\frac{\omega_1}{\omega_2}}\,L_1 C_o\left[I_2^2\left(\frac{K_3 - K_1}{\omega_2^{\,2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\,2} - \omega^2}\right)^2 - \frac{R^2\omega_1\omega_2 C_o^{\,2}}{D^2}\right]^{\frac{1}{2}} . \tag{230}$$

The formal existence proof will not be completed by the expansion of the Jacobian at the equilibrium point. It suffices to say that for proper tuning of the network of Figure 34 pairs of irrational frequency components are experimentally observed if the driving current is sufficiently large.

A physical interpretation of this "irrational response" with $\omega = \omega_1 + \omega_2$ is obtainable from a study of equations (219) and (220). Let it be assumed that small noise or transient voltages exist at the frequencies ω_1 and ω_2. Then the charge component at ω_1 will beat against the forced response at ω to produce a component at ω_2. Now if the existence conditions on tuning and driving current for $X_1 \neq 0$ and $X_2 \neq 0$ are satisfied, the modulation product at ω_2 has a greater amplitude than the original noise or transient amplitude. This larger component at ω_2 will now beat against the response at ω to produce a larger response at ω_1. The process is cumulative until X_1 and X_2 build up to their equilibrium conditions given by (223), (224), (226). Thus this class of response is essentially a self modulation process in which all power is drawn from the exciting source.

Similar "irrational responses" are possible for other relations between the frequencies of the responses and the exciting frequency. Assume that the nonlinear capacitor has a voltage-charge characteristic for which the highest term of its polynomial approximation is

$$\frac{D_{n-1}}{C_o} q^n \quad .$$

If $\omega_1 + \omega_2 = (n - 1)\omega$, the term of the second approximation which represents energy transfer from the harmonic response at ω to the response at ω_1, within a constant multiplier, is

$$\frac{nD_{n-1}}{C_o}(w_o + y_o)^{n-1}(y_1 + w_1) = \frac{nD_{n-1}}{C_o}\Big[w_o^{\,n-1} + (n-1)w_o^{\,n-2}y_o + - - - - - + (n-1)w_o y_o^{\,n-2} + y_o^{\,n-1}\Big](y_1 + w_1) \tag{231}$$

Now y_1 and w_1 contain harmonic response terms proportional to I sin θ. Expansion of the above relation would give the cross modulation terms showing energy interchange between ω_1, ω_2 and ω. However, since $w_o = X_1 \sin \psi_1$, $y_o = X_2 \sin \psi_2$, all terms of the above would contain powers of X_1 or X_2 higher than the first. Since the resistive loss terms vary as the first power of X_1 or X_2, it follows that as X_1 and X_2 approach zero together there is a point at which sufficient energy is no longer transferred from the driving force to sustain the X_1, X_2 response. Thus in the circuit of two degrees of freedom of Figure 34 "irrational responses" cannot build up from rest, for excitation of order ϵ , with $\omega_1 + \omega_2 = (n - 1)\omega$, and n greater than 2. This does not prohibit their existence but does say if they exist a transient excitation is necessary. The case of n = 2 was analyzed in detail above. The limitation on excitation magnitude is necessary since the analysis of this section has assumed a small excitation. With large driving forces it is possible for cross modulation terms $\omega \pm \omega_1$ and $\omega + \omega_2$ to appear in the first approximation.

The fact that $\omega_1 + \omega_2$ must be equal to ω, for these "irrational responses" to build up, has been experimentally verified. For circuits of the form of Figure 34 driven by a 6SJ7 tube with a current I $\doteq$ 10 milliamperes, it was easy to demonstrate irrational frequency responses if $\omega_1 + \omega_2 = \omega$, but no irrational responses were observed if $\omega_1 + \omega_2 = \omega(n - 1)$ with n greater than 2.

It is possible to obtain irrational responses, which build up from rest with $\omega_1 + \omega_2 = (n - 1)\omega$, $n > 2$, in circuits of three degrees of freedom, provided the third resonance occurs near the exciting frequency. In such circuits it is assumed that one resonant frequency falls near the excitation frequency ω and the others Ω_1 and Ω_2 near ω_1 and ω_2, where $\omega_1 + \omega_2 = (n - 1)\omega$. These circuits then possess solutions whose zeroth approximations are of the form

$$x_o = X_o \sin(\omega t + \phi_o),\ x_1 = X_1 \sin(\omega_1 t + \phi_1),$$

$$x_2 = X_2 \sin(\omega_2 t + \phi_2)$$

Then a cross modulation term of the form

$$nD_n X_o^{n-1} [\sin^{n-1}(\omega t + \phi_o)](x_1 + x_2)$$

will occur. That is, the $(n - 1)$-th harmonic of ω will beat against $X_1 \sin(\omega_1 t + \phi_1)$ and $X_2 \sin(\omega_2 t + \phi_2)$ to produce modulation products at the frequencies ω_2 and ω_1. Now since X_o^{n-1} does not approach zero with X_1 and X_2, it follows that the modulation cross products vanish as the first power of X_1 and X_2. The circuit loss terms are also proportional to the first powers of X_1 and X_2. Thus the circuit loss terms can remain less than the cross modulation terms as X_1 and X_2 approach zero together, and "irrational responses" can be made to build up from rest in triply resonant circuits with $\omega_1 + \omega_2 = (n - 1)\omega$. These "irrational solutions" will occur, of course, only for proper tuning, current input, and nonlinearity.

This section thus far has treated irrational responses that occur when the sum of ω_1 and ω_2 equals some harmonic of the excitation frequency ω. The response, when the difference of ω_1 and ω_2 equals ω or a harmonic thereof, is treated below. If $\Omega_1 - \Omega_2$ is approximately equal to ω and X_1, X_2 are small so that terms involving their second and higher powers are negligible, it can be shown for the circuit of Figure 34 that

$$w_o = X_1 \sin(\omega_1 t + \phi_1), \; yo = X_2 \sin(\omega_2 t + \phi_2)$$

and

$$w_1 = -\frac{I(K_2 - K_1)}{\epsilon(\omega_1^2 - \omega^2)} \sin\theta, \; y_1 = \frac{I(K_3 - K_1)}{\epsilon(\omega_2^2 - \omega^2)} \sin\theta \quad .$$

The equations which define the solutions to an order of ϵ^2 are

$$\frac{\partial^2 w_2}{\partial \theta^2} + \frac{\omega_1^2}{\omega^2} w_2 = -2A_2 \frac{\omega_1}{\omega^2} \cos(\frac{\omega_1}{\omega}\theta + \phi_1) \tag{232}$$

$$+ 2\sigma_2 \frac{\omega_1}{\omega^2} X_1 \sin(\frac{\omega_1}{\omega}\theta + \phi_1)$$

$$+ \frac{1}{\omega^2\epsilon^2}(\omega_1^2 - \Omega_1^2) X_1 \sin(\frac{\omega_1}{\omega}\theta + \phi_1)$$

$$- \frac{R K_2 X_1 \omega_1}{\epsilon^2\omega^2 L_1} \cos(\frac{\omega_1}{\omega}\theta + \phi_1)$$

$$- \frac{2 K_2 D}{\epsilon^2\omega^2 L_1 C_o}(w_o + y_o)(w_1 + y_1)$$

$$\frac{\partial^2 y_2}{\partial \theta^2} + \frac{\omega_2^{\ 2}}{\omega^2} = -2\, B_2 \frac{\omega_2}{\omega^2} \cos(\frac{\omega_2}{\omega}\theta + \phi_2) \tag{233}$$

$$+ 2\, \gamma_2 \frac{\omega_2 X_2}{\omega^2} \sin(\frac{\omega_2}{\omega}\, \theta + \phi_2) + \frac{X_2}{\varepsilon^2 \omega^2} (\omega_2^{\ 2} - \Omega_2^{\ 2}) \sin(\frac{\omega_2}{\omega}\, \theta + \phi_2)$$

$$- \frac{R\, K_3\, \omega_2\, X_2}{\varepsilon^2\, \omega^2\, L_1} \cos(\frac{\omega_2}{\omega}\, \theta + \phi_2) + \frac{2\, K_3\, D}{\varepsilon^2\, \omega^2\, L_1 C_o} (w_o + y_o)(w_1 + y_1)\ .$$

Since $\omega_1 - \omega_2$ is assumed equal ω, the equilibrium conditions which determine X_1, X_2 become

$$0 = R\, X_1\, \omega_1 - \frac{D\, X_2\, I}{C_o}\left[\frac{K_3 - K_1}{\omega_2^{\ 2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\ 2} - \omega^2}\right] \cos(\phi_1 - \phi_2) \tag{234}$$

$$0 = R\, X_2\, \omega_2 + \frac{D\, X_1\, I}{C_o}\left[\frac{K_3 - K_1}{\omega_2^{\ 2} - \omega^2} - \frac{K_2 - K_1}{\omega_1^{\ 2} - \omega^2}\right] \cos(\phi - \phi_2)\ . \tag{235}$$

There is no value of $\cos(\phi_1 - \phi_2)$ for which these conditions are simultaneously satisfied. Thus it is concluded that irrational responses do not exist in circuits of two degrees of freedom if $\omega_1 - \omega_2 = \omega$. A similar proof could be given to show that such responses do not occur in circuits of three degrees of freedom if $\omega_1 - \omega_2$ equals a harmonic of the excitation frequency.

This section has shown analytically that pairs of charge components with frequencies ω_1 and ω_2 can exist and build up from rest in doubly resonant circuits if $\Omega_1 + \Omega_2$ is approximately equal to ω; or they can exist in triply resonant circuits if $\Omega_1 + \Omega_2 \doteq n\omega$ and one resonant frequency is near ω. No such responses exist if the difference of Ω_1 and Ω_2 is near ω or a harmonic thereof.

The circuit of Figure 35 was used experimentally to investigate the property of the irrational solutions that occur when $\omega_1 + \omega_2 = \omega$. Figure 36 shows oscilloscope pictures of waveforms of the voltage observed across C with the circuit above for two values of e_s. An electronic switch was used to display the input signal e_s as the upper trace. In each case the oscilloscope sweep was synchronized at the frequency of e_s. The plate network was resonant at 4 and 16 KC, and the input frequency was 20 KC. Figure 36 A shows that with $e_s = 3.0$ volts there exist stable subharmonics at 1/5 and 4/5 the driving frequency. In Figure 36 B with $e_s = 2.0$ the response frequencies are irrational.

The plate network of Figure 35 was retuned to resonate at 3.3 and 7.7 kilocycles to avoid subharmonic responses with an exciting of 11 kilocycles. The principal amplitudes of the currents through the nonlinear capacitor were then measured as a function of alternating grid voltage, e_s. The results of this test are given in Table 10. The response frequencies are the excitation frequency, $f = 11$ kc, and approximately 3.3 and 7.7 kc which are denoted f_2 and f_1 respectively. The frequency f_1 changed by approximately 9 cycles between the lowest and highest values of e_s.

The amplitudes of the currents at f_1 and f_2 are plotted as a function of e_s in Figure 37. It is noted that the ratios of these current amplitudes are approximately constant as predicted by the analysis.

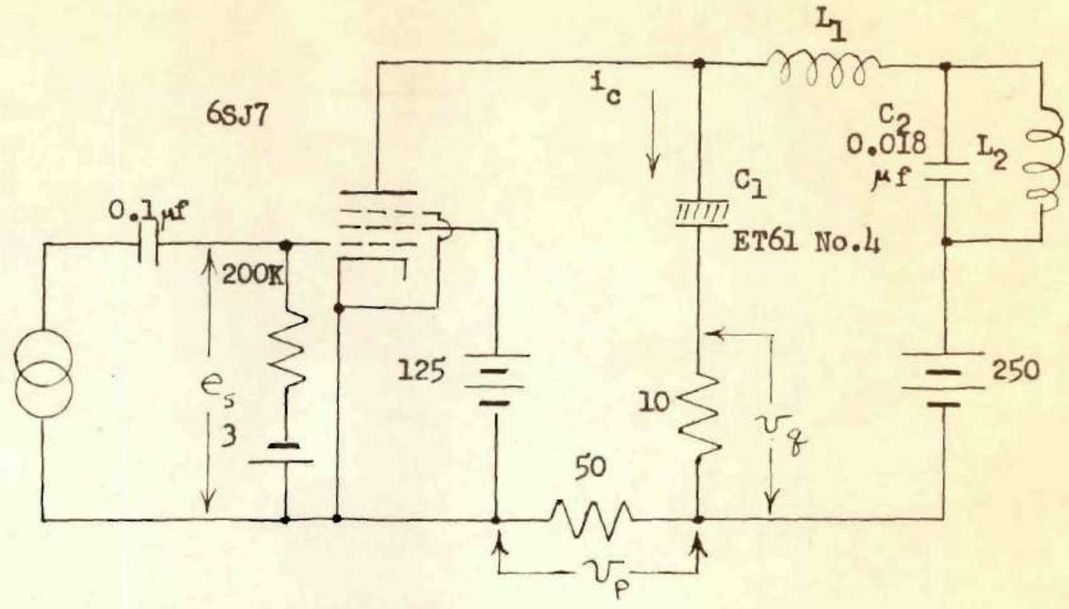


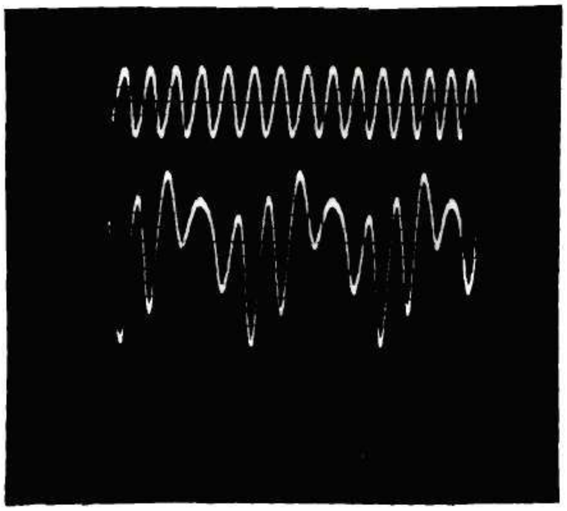
FIG. 35. DOUBLY RESONANT EXPERIMENTAL CIRCUIT

Table 10. Irrational Response of Figure 35 Circuit

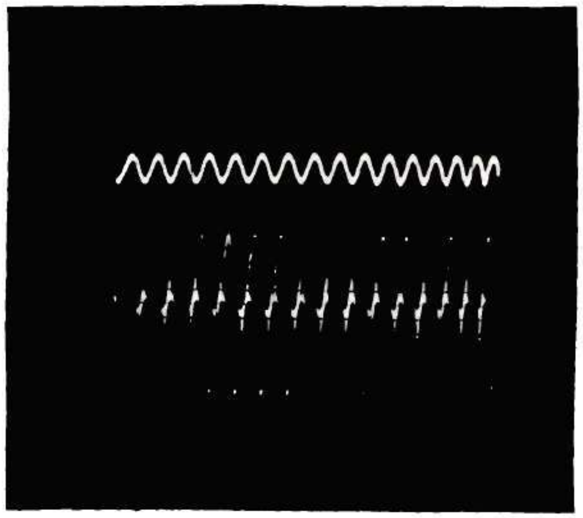
$f = 11kc$, $f_1 = 7.7kc$, $f_2 = 3.3kc$, $L_1 = 179mh$, $L_2 = 104mh$, 86^oF

$V_c = I_c R$ in rms volts

e_s at ω	at ω_2	at ω_1	at ω	V_p at ω
1.45	0	0	.030	.085
1.50	.0015	.0047	.031	.088
1.60	.0042	.0122	.0325	.094
1.80	.0078	.023	.036	.108
2.0	.0118	.034	.040	.118
2.2	.0144	.043	.044	.127
2.4	.017	.053	.048	.137
2.6	.020	.060	.053	.147



A. SUBHARMONIC
$e_s = 3.0$



B. IRRATIONAL
$e_s = 2.0$

FIG. 36. OSCILLOSCOPE WAVEFORMS OF SUBHARMONIC AND IRRATIONAL VOLTAGES

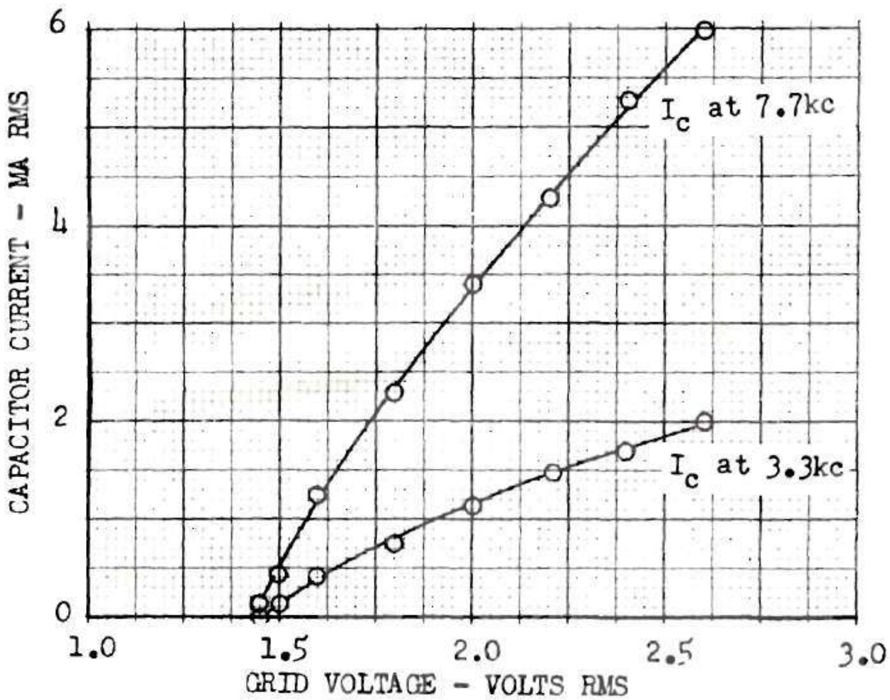


FIG. 37. IRRATIONAL RESPONSE AS A FUNCTION OF GRID VOLTAGE

If a second audio oscillator whose frequency is near $\Omega_2/2\pi$ is applied to the grid of the tube-control in Figure 35, it is possible to cause the frequency of the capacitor current at f_2 to synchronize with this oscillator frequency. That is, irrational frequencies can be synchronized with the frequency of small injected signals as in free running oscillator circuits. This fact suggests that it is possible to utilize these irrational responses to make subharmonics of high order self starting. In the next section this possibility is studied in detail.

Subharmonics of Third and Higher Orders in Doubly Resonant Circuits

In this section the properties of subharmonics of greater than second order will be studied. In particular, the conditions are derived under which the irrational responses of the previous section can be used to cause a subharmonic to build up from rest. Consideration is given to the determination of conditions under which a circuit possesses a stable pair of subharmonic frequencies or a pair of incommensurate frequencies. For convenience this section deals only with a network with two resonant frequencies as shown in Figure 34.

General subharmonic initiation conditions.--In the previous section of this chapter it was shown that circuits with resonant frequencies Ω_1 and Ω_2 could possess a response that builds up from rest, provided $\Omega_1 + \Omega_2$ is approximately equal to the excitation frequency. Experimentally it can easily be shown that such responses are possible whether or nor Ω_1 is near a subharmonic of ω. Analysis of such responses by any of the methods of Chapter II necessitates that the frequencies of the response be either subharmonic or well removed from the tuning condition for subharmonic resonance. This is a limitation of analysis, and it is not a physical limitation on circuit performance. The analytical difficulty exists because in each method of solution of quasilinear differential equations the solution is made periodic by equating to zero all terms, including losses, of its right hand side which occur at the natural frequency of the lossless circuit. Thus, if a circuit possesses responses at frequencies Ω_1, Ω_2 which are nearly but not exactly subharmonics, these terms would form combination frequencies that would differ from Ω_1 and Ω_2 by small quantities. If an analytical solution of such a problem is attempted, the response at Ω_1 and Ω_2 could be periodic, but there would be terms whose frequencies differ slightly from Ω_1 and Ω_2. These terms would have amplitudes which approach infinity as their frequencies approach Ω_1 or Ω_2. This difficulty can be easily seen from equation (210) of the irrational analysis, if one lets Ω_1 approach either four Ω_2 or the excitation frequency ω. This type of limitation of presently available analysis is commonly encountered, and is referred to as the problem of small divisors. This limitation prevents general analytical solution of quasilinear circuits which are driven by generators whose frequencies are unrelated.[61] Thus, the methods of analytical approximation fail to yield meaningful results when there exist in the solution two or more combination- or response-frequencies which differ by a small quantity.

The experimentally proved existence of irrational solutions even near a subharmonic resonance suggests that if such a response is allowed to build up to equilibrium it can become synchronized at a subharmonic of the exciting frequency. That is, if the network is tuned to favor subharmonic resonance, as the response builds up from rest, the nonlinearity will produce combination frequencies or modulation products which will tend to cause the response to synchronize at a subharmonic of the exciting frequency. That this can occur is shown by Figures 36 A, B, of the previous section, which show that either irrational or subharmonic responses can be obtained as the driving voltage amplitude is varied. Such a synchronization is similar to the subharmonic synchronization of free running oscillators which has been studied by Cohen(62) and others.

Now if irrational solutions can be made to build up from rest conditions and these become stable subharmonics upon reaching equilibrium, this process can be made to render high order subharmonics self starting. This is the principle behind Manley's patent[8]. It is emphasized that as the response starts to build up it may or may not be subharmonic. A study of subharmonic response in circuits of two degrees of freedom then divides into two parts. One part investigates the condition under which a response will build up. The other determines whether such a response is subharmonic, that is synchronized, or irrational. The remainder of this paragraph will deal with the conditions under which a response near the subharmonic is excited from rest. It is well to note that since this section is limited to circuits of two degrees of freedom only the second order curvature is used to excite the response from rest but the non-linear characteristic must have a term which varies as q^n if a stable nth order subharmonic is to exist. Thus circuits of three degrees of freedom in which $\Omega_1 + \Omega_2 = (n - 1)\omega$ would be more apt to have a stable subharmonic of order n, if any response at Ω_1 is excited, since the same degree of non-linearity of the characteristic is necessary for excitation and synchronization. The circuit of two degrees of freedom favors easier excitation conditions and it is simpler. These are the reasons it is treated in detail here.

Conditions under which a response near a subharmonic of order n can build up from rest in the circuit of Figure 34 can be derived from the irrational solutions of the previous paragraph. Due to the "problem of small divisors," it is necessary to assume that the frequencies ω_1 and ω_2 are subharmonic. It is clear that often this will not be true, but ω_1 and ω_2 will differ only slightly from subharmonic so that initiation conditions for responses near these subharmonics can be derived on the assumption that these are subharmonic. It is assumed that Ω_1 is near $(n - 1)\frac{\omega}{n}$ and Ω_2 is near $\frac{\omega}{n}$. The subharmonic frequencies are $\omega_2 = \frac{\omega}{n}$ and $\omega_1 = (n - 1)\frac{\omega}{n}$, since $\omega_1 + \omega_2 = \omega$. It is also assumed that X_1 and X_2 are small, which is certainly true near their threshold, so that only the terms containing first powers of X_1 and X_2 need be retained. The characteristic of the nonlinear capacitor is assumed to be

$$\frac{1}{C_1} = \frac{1}{C_o} (1 + D_1 q + - - - + D_n q^n) \quad ,$$

where

$$\frac{D_1}{L_1 C_o}$$

is of order ϵ, and all higher coefficients are assumed of order ϵ^2. Then if the losses and detuning are of order ϵ^2, the Kryloff second approximation analysis as used in the previous section gives the conditions which define X_1, X_2, ϕ_1, ϕ_2 for small non-zero X_1, X_2, as

$$\frac{dX_1}{dt} = -\frac{K_2RX_1}{2L_1} + \frac{nDIK_2X_2}{2(n-1)\omega^3L_1C_o}\left[\frac{K_3-K_1}{1-\frac{1}{n^2}} - \frac{K_2-K_1}{1-(1-\frac{1}{n})^2}\right]\cos(\phi_1+\phi_2) = 0 \qquad (236)$$

$$\frac{d\phi_1}{dt} = -\frac{n}{2(n-1)\omega}\left[\omega^2(1-\frac{1}{n})^2 - \Omega_1^2\right] - \frac{nDIK_2X_2}{2(n-1)\omega^3L_1C_oX_1}\left[\frac{K_3-K_1}{1-\frac{1}{n^2}} - \frac{K_2-K_1}{1-(1-\frac{1}{n})^2}\right]\sin(\phi_1+\phi_2) = 0 \qquad (237)$$

$$\frac{dX_2}{dt} = \frac{RK_3X_2}{2L_1} - \frac{nDIK_3X_1}{2\omega^3L_1C_o}\left[\frac{K_3-K_1}{1-\frac{1}{n^2}} - \frac{K_2-K_1}{1-(1-\frac{1}{n})^2}\right]\cos(\phi_1+\phi_2) = 0 \qquad (238)$$

$$\frac{d\phi_2}{dt} = -\frac{n}{2\omega}\left[\frac{\omega^2}{n^2} - \Omega_2^2\right] + \frac{nDK_3IX_1}{2n\omega^3L_1C_oX_2}\left[\frac{K_3-K_1}{1-\frac{1}{n^2}} - \frac{K_2-K_1}{1-(1-\frac{1}{n})^2}\right]\sin(\phi_1+\phi_2) = 0 \qquad (239)$$

These equations possess a simultaneous solution only if zero detuning exists, so that $\Omega_2 = \omega_2 = \frac{\omega}{n}$ and $\Omega_2 = \omega - \omega_1 = \omega(1 - \frac{1}{n})$. This is a result of the fact that for X_1 and X_2 very small there is no assurance that the solution is subharmonic and not irrational.

For the zero detuning, cos $(\phi_1 + \phi_2)$ is unity and the equilibrium conditions reduce to

$$0 = RX_1 - \frac{nDIX_2}{(n-1)\omega^3 C_o}\left[\frac{K_3 - K_1}{1 - \frac{1}{n^2}} - \frac{K_2 - K_1}{1 - (1 - \frac{1}{n})^2}\right] \tag{240}$$

and

$$0 = RX_2 - \frac{nDIX_1}{\omega^3 C_o}\left[\frac{K_3 - K_1}{1 - \frac{1}{n^2}} - \frac{K_2 - K_1}{1 - (1 - \frac{1}{n})^2}\right] . \tag{241}$$

These are satisfied if

$$R^2 = \frac{n^2 D^2 I^2}{(n-1)\omega^6 C_o{}^2}\left[\frac{K_3 - K_1}{1 - \frac{1}{n^2}} - \frac{K_2 - K_1}{1 - (1 - \frac{1}{n})^2}\right]^2 . \tag{242}$$

If R^2 is less than the quantity on the right hand side of (241), X_1 and X_2 may not be zero. The response can then build up from the rest condition $X_1 = X_2 = 0$. Thus responses at frequencies near $\frac{\omega}{n}$ and $(n-1)\frac{\omega}{n}$ can build up from rest provided

$$\frac{\omega^2}{n^2} - \Omega_2^2 \text{ and } (1 - \tfrac{1}{n})^2\omega^2 - \Omega^2$$

are both very small and

$$\frac{nDI}{(n-1)\omega^3 C_o}\left|\frac{K_3 - K_1}{1 - \frac{1}{n^2}} - \frac{K_2 - K_1}{1 - (1 - \frac{1}{n})^2}\right| > R . \tag{243}$$

This analysis does not yield a measure of the detuning allowable. The relations above thus allow one to determine if it is possible with a given loss and nonlinearity to excite a response near the n-th subharmonic frequency.

The problem of determining if a solution leads to irrational or subharmonic frequencies could be investigated by studying the conditions under which a solution computed under the assumption that it is a subharmonic is stable. If the solution has a stable phase, it is a subharmonic. Otherwise the solution is either completely unstable or an irrational solution. If the X_1, X_2, ϕ_1, ϕ_2 which satisfy the equiliibrium conditions

$$\begin{aligned}
\dot{X}_1 &= P_1(X_1, X_2, \phi_1, \phi_2) = 0 \\
\dot{X}_2 &= P_2(X_1, X_2, \phi_1, \phi_2) = 0 \\
\dot{\phi}_1 &= S_1(X_1, X_2, \phi_1, \phi_2) = 0 \\
\dot{\phi}_2 &= S_2(X_1, X_2, \phi_1, \phi_2) = 0
\end{aligned} \tag{244}$$

are perturbed by the small quantities α_1, α_2, η_1, η_2, the first order variational equations are

$$\frac{d\alpha_1}{dt} = \frac{\partial P_1}{\partial X_1}\alpha_1 + \frac{\partial P_1}{\partial X_2}\alpha_2 + \frac{\partial P_1}{\partial \phi_1}\eta_1 + \frac{\partial P_2}{\partial \phi_2}\eta_2 \tag{245}$$

$$\frac{d\alpha_2}{dt} = \frac{\partial P_2}{\partial X_1}\alpha_1 + \frac{\partial P_2}{\partial X_2}\alpha_2 + \frac{\partial P_2}{\partial \phi_1}\eta_1 + \frac{\partial P_2}{\partial \phi_2}\eta_2$$

$$\frac{d\phi_1}{dt} = \frac{\partial S_1}{\partial X_1}\alpha_1 + \frac{\partial S_1}{\partial X_2}\alpha_2 + \frac{\partial S_1}{\partial \phi_1}\eta_1 + \frac{\partial S_1}{\partial \phi_2}\eta_2$$

$$\frac{d\phi_2}{dt} = \frac{\partial S_2}{\partial X_1}\alpha_1 + \frac{\partial S_2}{\partial X_2}\alpha_2 + \frac{\partial S_2}{\partial \phi_1}\eta_1 + \frac{\partial S_2}{\partial \phi_2}\eta_2 .$$

The partial derivatives are evaluated at the equilibrium values of X_1, X_2, ϕ_1, ϕ_2. Stability properties are determined from the roots of

$$\begin{vmatrix} \frac{\partial P_1}{\partial X_1} - P & \frac{\partial P_1}{\partial X_2} & \frac{\partial P_1}{\partial \phi_1} & \frac{\partial P_2}{\partial \phi_2} \\ \frac{\partial P_2}{\partial X_1} & \frac{\partial P_2}{\partial X_2} - P & \frac{\partial P_2}{\partial \phi_1} & \frac{\partial P_2}{\partial \phi_2} \\ \frac{\partial S_1}{\partial X_1} & \frac{\partial S_1}{\partial X_2} & \frac{\partial S_1}{\partial \phi_1} - P & \frac{\partial S_1}{\partial \phi_2} \\ \frac{\partial S_2}{\partial X_1} & \frac{\partial S_2}{\partial X_2} & \frac{\partial S_2}{\partial \phi_1} & \frac{\partial S_2}{\partial \phi_2} - P \end{vmatrix} = 0 \qquad (246)$$

A solution is then subharmonic, irrational, or completely unstable as the real parts of the roots of the characteristic determinant are negative, zero, or positive. In order that a solution be irrational, one root must have zero real part for a range of input voltage or frequency. This follows since any solution will have at least a single point for which the root has zero real part corresponding to the transition from stable to unstable conditions.

The equilibrium equations (244) for circuits of two degrees of freedom can be found by the Kryloff second approximation. Unfortunately these conditions are of such algebraic complexity, for third or higher order subharmonics, that solutions can be calculated only numerically. Since the stability conditions require a knowledge of the equilibrium values of X_1, X_2, ϕ_1, ϕ_2, the ranges of input voltages and frequencies over which the solution is a stable subharmonic can be obtained by the analytical approximation methods only by repeated calculations. The next paragraph gives some experimental results for third and higher order circuits.

<u>Third and higher order subharmonics</u>.--The circuit of Figure 35, with the ET61 No. 3 Capacitor as C_1, E_b and E_{c_2} of 250, and E_{c_1} of -5 volts, was used to generate subharmonics of third and higher orders. In each set of tests L_1 and L_2 were varied to resonate the plate network at angular frequencies Ω_1 and Ω_2 which are approximately rational fractions of the driving frequency ω and such that $\Omega_2 = \omega - \Omega_1$. Initial tuning was accomplished by varying L_1 and L_2 to maximize the ac plate voltage with the input signal at first Ω_2 and then Ω_1. The final tuning in each case was varied slightly to maximize the range over which the subharmonic was stable. It was found possible in the case of third and fourth subharmonics to tune the network so that only the stable subharmonics appeared. That is, in these cases the ranges of input drive and frequency, over which the solutions were stable, were equal to the ranges over which any response could build up from rest. Thus no irrational solutions were observed for these tuning conditions.

Amplitude response and tuning data were taken with the plate network resonating at about 4 and 8 kc, or $\frac{1}{3}$ and $\frac{2}{3}$ the driving frequency of 12 kc. The measurements of circuit response under detuned conditions were made by varying the exciting frequency about its reference value of 12 kc. Thus the frequency deviation from the sum of the two resonant frequencies is $\Delta F = f - 12{,}000$. Capacitor currents and voltages at the frequencies of 12, 8 and 4 kilocycles were measured for various amplitudes of 12 kc grid voltage for the zero detuning condition. The results of these measurements are given in Table 11. Figure 38 is a plot of capacitor current at 4 and 8 kc as a function of alternating grid voltage. Figure 39 gives curves of capacitor current as a function of ΔF for a constant grid voltage of 3 volts rms.

The simple subharmonics of orders third, fourth, and up to the tenth were generated in the above circuit. In each case a response was made to build up from rest by tuning the network so that the sum of its two resonant frequencies was approximately equal to the exciting frequency. Thus the fourth order subharmonic of a 16 kilocycle excitation was obtained with the network resonant at 4 and 12 kilocycles. For this particular tuning condition, the response, for a 3.8 volt rms grid voltage, was a stable subharmonic for frequency deviations from 16 kc between + 26 and - 320 cycles. Outside this frequency interval irrational frequencies exist for frequency deviations from 16 kc for variations between + 114 and - 922 cycles; beyond these points only the 16 kc harmonic response was observed. The fifth subharmonic with a 4-volt grid voltage at 20 kc was stable over a frequency range of less than 100 cycles variation of the exciting frequency. An irrational response existed for frequency deviations from 20 kc from + 350 to - 600 cps. As the subharmonic order n increases, the higher resonant frequency Ω_1 approaches the excitation frequency, since $\Omega_1 = \omega - \Omega_2 = \omega(1 - \frac{1}{n})$. It was observed that as the subharmonic order increased, the input voltage and frequency range over which stable subharmonics exist became very small. However, it is easier to excite a response as Ω_1 approaches ω and irrational responses exist for wide ranges of input grid voltage and frequency. A subharmonic of

Table 11. Third Order Subharmonic Response

$f = 12$ kc, $f_1 = 8$ kc, $f_2 = 4$ kc
Circuit of Figure 35

e_s	I_c - ma			Voltage across C.		
at w	w	2/3 w	w/3	w	2/3 w	w/3
2.07	7.9		0	58.6	0	0
2.15	8.4	2.7	2.0	61.0	28.6	45.1
2.50	9.5	4.7	3.7	69.7	50.8	85.0
3.00	11.4	7.6	5.5	79.4	85.8	126
3.80	13.3	13.3	6.4	101.5	146	162

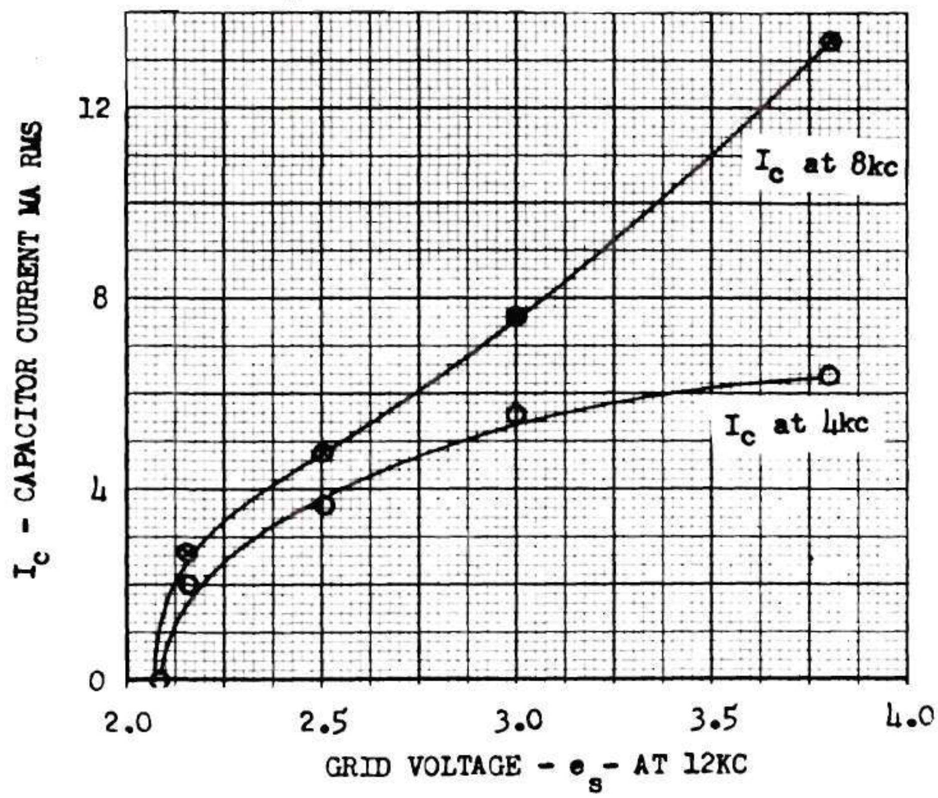


FIG. 38. THIRD ORDER SUBHARMONIC RESPONSE

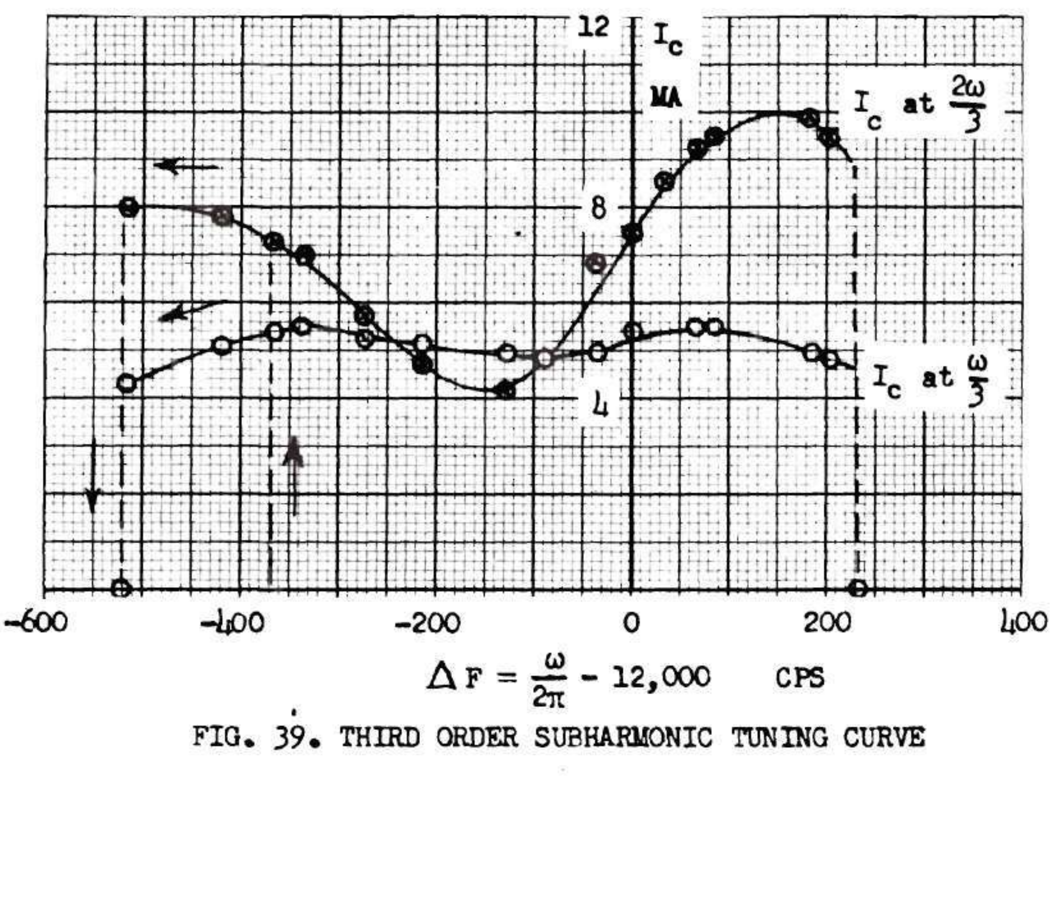


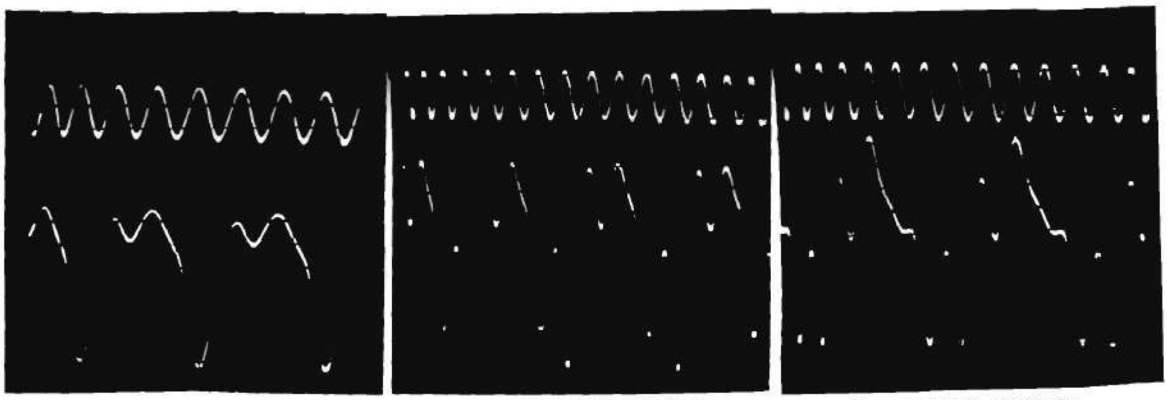
FIG. 39. THIRD ORDER SUBHARMONIC TUNING CURVE

seventeenth order, $\omega_1 = \frac{\omega}{17}$, was the highest frequency division obtained. This response was very marginally stable.

It is noted that subharmonics other than simple fractions can be excited for proper tuning conditions of the network. That is, subharmonics of frequency $\frac{r}{s}\omega$ can be excited with r and s integers greater than one. Such responses, of course, exist in pairs because of the network tuning condition $\omega_1 = \omega - \omega_2$. Some of these rational fraction responses observed were 3 and 7 kc with a 10 kc excitation, 2 and 5 kc with 7 kc excitation, and 4 and 7 kc with 11 kc excitation. With the circuit of Figure 35 it was possible to generate with a 10 kc exciting frequency the following pairs of frequencies: 1, 9 kc; 2, 8kc; 3, 7; 4, 6 and 5, 10 kcs. Thus by merely retuning the network all subharmonics of 10 kc which are multiples of 1 kc can be obtained.

Figures 40A through 40H are oscilloscope pictures of the waveform of plate voltage for subharmonics of the third through the tenth orders. In each case where two traces occur the upper trace is the driving voltage. The two traces were obtained by using a two-channel electronic switch. Figures 41A through 41C show waveforms of the rational fractional subharmonics whose frequencies are $\frac{3}{10}$, $\frac{7}{10}$, $\frac{2}{9}$, $\frac{7}{9}$, $\frac{2}{5}$, $\frac{3}{5}$ of the excitation frequency. Figures 41D through 41H show waveforms of subharmonics whose frequencies are rational fractions of the excitation frequency and whose some equals the second harmonic of the excitation, or $\omega_1 + \omega_2 = 2\omega$. These waveforms were obtained with a relatively high current from a 6SJ7 tube.

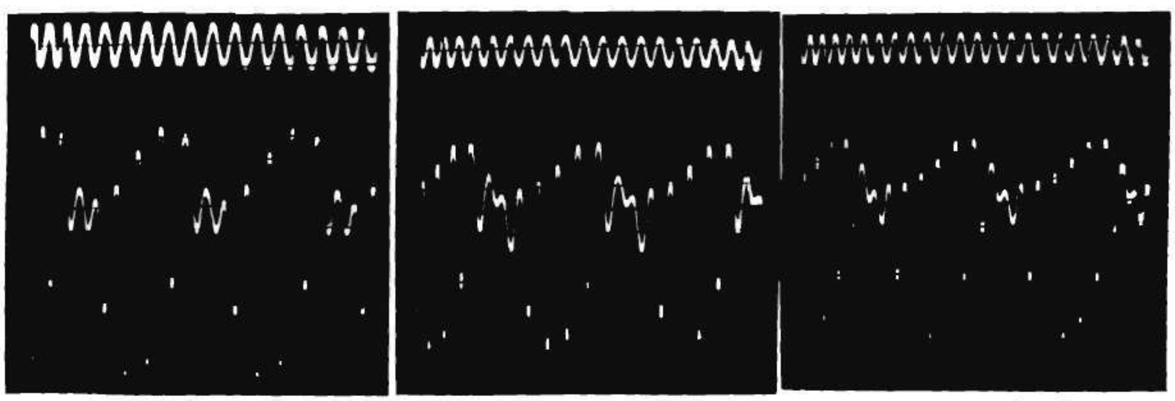
A triply resonant circuit was used as a plate network for the 6SJ7 tube. Figure 41I whows the waveform observed when the circuit was tuned to 8, 5, and 3 kilocycles and driven from a 16 kc audio



A. THIRD ORDER
$f = 12, f_1 = 8, f_2 = 4$kc

B. FOURTH ORDER
$f = 16, f_1 = 12, f_2 = 4$

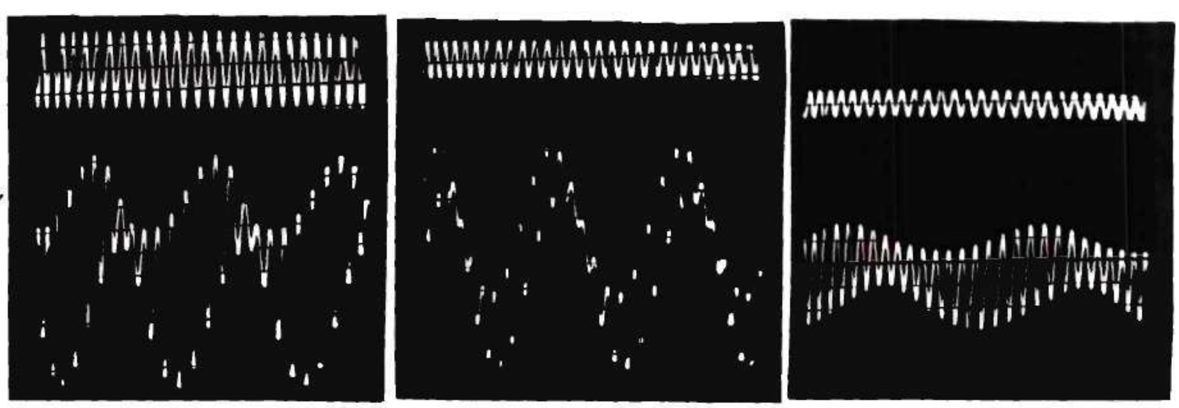
C. FIFTH ORDER
$f = 20, f_1 = 16, f_2 = 4$



D. SIXTH ORDER
$f = 18, f_1 = 15, f_2 = 3$

E. SEVENTH ORDER
$f = 21, f_1 = 18, f_2 = 3$

F. EIGHTH ORDER
$f = 24, f_1 = 21, f_2 = 3$

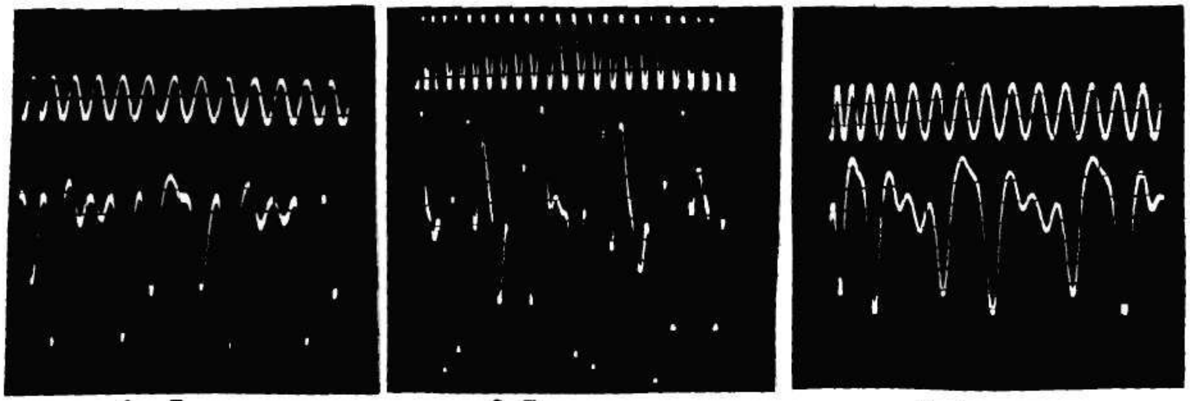


G. NINTH ORDER
$f = 27, f_1 = 24, f_2 = 3$

H. TENTH ORDER
$f = 30, f_1 = 27, f_2 = 3$

I. FIFTEENTH ORDER
$f = 45, f_1 = 42, f_2 = 3$

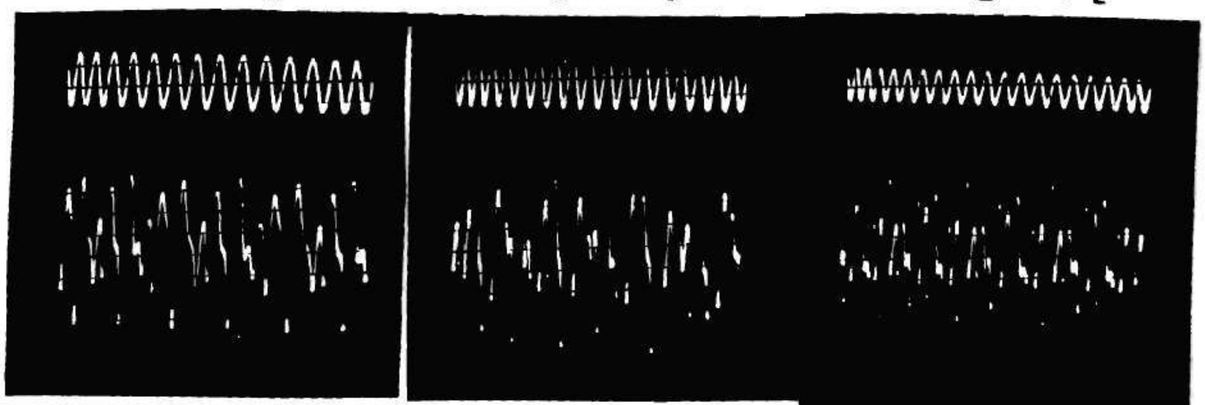
FIG. 40. SUBHARMONIC WAVEFORMS



A. $\frac{3}{10}, \frac{7}{10}$ RESPONSE
$f = 10, f_1 = 7, f_2 = 3$ kc

B. $\frac{2}{9}, \frac{7}{9}$ RESPONSE
$f = 9, f_1 = 7, f_2 = 2$

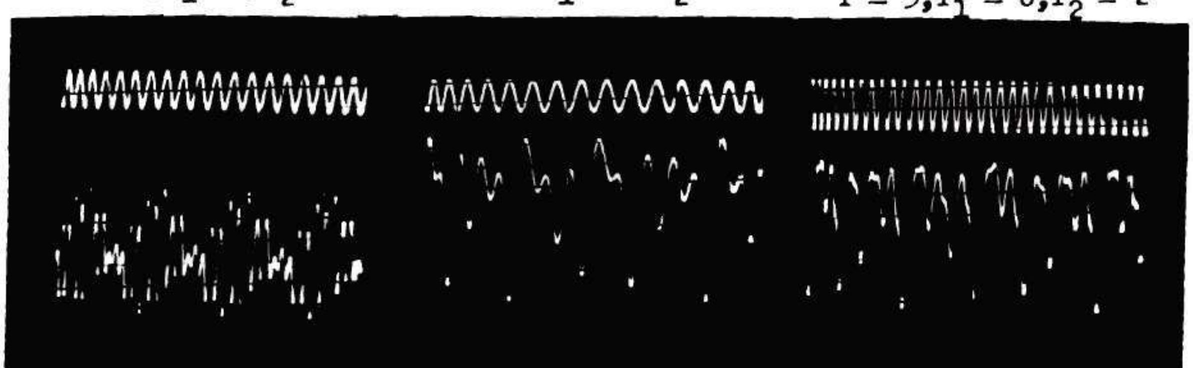
C. $\frac{2}{5}, \frac{3}{5}$ RESPONSE
$f = 10, f_1 = 6, f_2 = 4$



D. $\frac{4}{5}, \frac{6}{5}$ RESPONSE
$f = 5, f_1 = 6, f_2 = 4$

E. $\frac{3}{5}, \frac{7}{5}$ RESPONSE
$f = 5, f_1 = 6, f_2 = 4$

F. $\frac{2}{5}, \frac{8}{5}$ RESPONSE
$f = 5, f_1 = 8, f_2 = 2$



G. $\frac{1}{5}, \frac{9}{5}$ RESPONSE
$f = 10, f_1 = 18, f_2 = 2$

H. $1, \frac{5}{8}, \frac{3}{8}$ RESPONSE
$f = f_1 = 8, f_2 = 5, f_3 = 3$ kc
TRIPLE RESONANCE

I. $\frac{1}{2}, \frac{5}{16}, \frac{3}{16}$ RESPONSE
$f = 16, f_1 = 8, f_2 = 5, f_3 = 3$ kc
TRIPLE RESONANCE

FIG. 41. SUBHARMONIC WAVEFORMS

oscillator. This waveform contains frequencies which are $\frac{1}{2}$, $\frac{5}{16}$, and $\frac{3}{16}$ of that of the excitation. In this case the second subharmonic is used to cause the two lower-frequency components to build up from rest, since $\omega_1 = \omega_2 + \omega_3$. This is one example of the classes of responses possible in more complex networks.

This section has shown experimentally that it is possible to cause high-order subharmonics to be self starting. It has shown that if the plate network resonates at Ω_1 and Ω_2 which are near rational fractions of ω, a response can build up from rest conditions. Although no general analysis has been given, it has been shown experimentally that the response is subharmonic over a range of conditions and irrational outside this range. However, it is possible with present ferroelectric dielectrics to obtain second, third or fourth order subharmonic responses without irrational frequencies occurring if the excitation amplitude is varied.

Chapter 6. CONCLUSIONS

Three analytical approximation methods, which are applicable to the study of subharmonic resonance, were developed in Chapter II. The perturbation series and Kryloff successive approximation methods are capable of yielding a second or higher approximation to the solution of a quasilinear differential equation. The method of equivalent linearization yields a first approximate solution for small excitation. It was shown that circuits of low loss and small nonlinearity having n degrees of freedom can be analyzed by reduction of the circuit equations to n second-order quasilinear equations. A formal method of constructing a second approximation to the solution of a circuit of two degrees of freedom is presented in the Appendix.

The power factor and nonlinearity of a number of ceramic ferroelectric samples were measured to determine their usefulness as nonlinear circuit elements. Data on the three dielectrics which exhibited greatest nonlinearity were given in Chapter III. Several methods of measurement of the nonlinear characteristics of the dielectrics were used, including dynamic charge versus voltage hysteresis loops. In the most accurate method of determining the coefficients of a polynomial to approximate the capacitor charge voltage characteristic the amplitudes of the harmonic components of capacitor charge were measured when a sine wave of charge flowed through the capacitor. The ET61 ferroelectric of D. M. Steward Company was the most useful in subharmonic circuits. For this dielectric, of 0.02 inch thickness, the incremental capacity with 300 volts bias is about one-fourth its incremental capacity at zero bias. Its power factor is about 0.05 when unbiased. The voltage characteristic as a function of charge of nonlinear dielectrics can be approximated by a polynomial of charge. The coefficients of the polynomial approximation vary if sizeable variations occur in the amplitude of the charge. This is a result of the dependence of the dielectric on previous history.

In Chapter IV it was shown that a second order subharmonic solution for a single degree of freedom circuit derived by the Kryloff approximation method yields a result similar to that obtained by perturbation series. The results differ only as one-half the input excitation frequency is unequal to the circuit resonant frequency. A similar difference exists between third order subharmonic solutions computed by Duffing's iteration method and the Kryloff approximation. Analytically and experimentally it has been shown that the second subharmonic in a single loop circuit will build up from rest conditions. If one-half the excitation frequency is less than the circuit resonant frequency, the current or voltage to initiate the subharmonic is greater than that required to sustain it once established. Single loop circuits are adequate to generate second subharmonics of excitation frequencies up to a few hundred kilocycles, and circuits with resonant input loops seem capable of frequency division at frequencies to perhaps 20 megacycles.

The growth and decay behavior of single loop second subharmonics has been studied by observing envelope waveforms when the excitation is gated. Envelope waveform pictures of the subharmonic response, when the excitation is sinusoidally amplitude-modulated, show that if the modulation frequency is sufficiently high and the subharmonic envelope is nearly a replica of the excitation envelope. Envelope distortion increases as the modulation frequency decreases, and the subharmonic vanishes if the input carrier amplitude is too low or the modulation index too high.

It is possible to analytically predict the phenomena observed in principal or subharmonic resonance of circuits containing ferroelectric capacitors. However, it is necessary to carefully restrict the range of variables used in experimental work to that assumed in the analytical treatment. The analytical methods used herein all assume that nonlinear, loss and detuning terms are of the order of a power of a small parameter. Thus to obtain a complete analytical treatment of a particular problem, it is necessary to analyze the special cases as each of the loss, detuning and nonlinearity terms takes on all significant orders of the small parameter. That is, if it is desired to analyze in detail the second order subharmonic resonance of a single loop circuit, a solution can be derived by the Kryloff second approximation for special cases. The quasilinear circuit is assumed to have a second degree nonlinearity of order ϵ. Then analysis is needed for the cases where the excitation is of order ϵ^0 and ϵ, loss of order ϵ and ϵ^2, detuning order ϵ and ϵ^2, undesired nonlinear terms of order ϵ and ϵ^2, and combination thereof. In a specific case not all combinations need be analyzed since usually some cases will yield trivial results or results obtainable by deduction from another case. The second approximation analysis of the second subharmonic with detuning of order ϵ^2 fails to predict a lower limit to the excitation frequency for which the subharmonic exists. However, a lower limit could be calculated, if the detuning were assumed of order ϵ, in a second approximation solution.

Comparison of experimental and analytical numerical results shows only fair agreement between the amplitudes of the subharmonics although the class of behavior is satisfactorily predicted by analysis. The differences between computed and measured results were in each case traceable to either a variable exceeding the order of ϵ assumed in the analysis, or errors in tuning, or an inadequate representation of the nonlinear characteristic by the polynomial approximation. It was difficult to accurately tune low-loss circuits because of variations of the capacitor characteristic with applied voltage or charge. Heating of the ferroelectric dielectric causes erratic behavior near critical or transition points of the response.

A second approximation solution for excitation of the order of unity can be developed. The third order subharmonic first approximation solution, with excitation of order unity, is derived in Chapter IV. Third and higher order subharmonics can exist but will not build up from rest conditions in singly resonant circuits.

It was shown in Chapter V that responses in multiple loop circuits can exist with frequencies which are not rational fractions of the driving frequency, and these response frequencies vary with excitation amplitude. This class of response can be utilized to cause currents which are nearly subharmonic to build up from rest; under equilibrium conditions these may synchronize and become stable subharmonics. Such conditions will generally lead to a response having a harmonic, an irrational, and a subharmonic region with input signal or frequency variations. However, with the ET61 dielectric it was possible to generate stable third and fourth subharmonics, which exist over all the region where a response other than harmonic exists. Analytical conditions for the build-up of responses near subharmonic were derived. However, analysis has not yet been successful in determining a region over which the response is a stable subharmonic.

It is suggested that some of the approximate methods of treating synchronization be considered in analyzing this problem. A solution for the subharmonic of order n can be derived by the Kryloff approximation method, but the subharmonic amplitude can be found explicitly only numerically because of the algebraic complexity of the equilibrium conditions.

Second-order subharmonics are easily excited in doubly resonant circuits, either voltage- or current-fed, in which one resonant frequency is equal to the excitation frequency and the other one-half the excitation frequency. If a significant third-degree curvature exists in the characteristic of the nonlinear element, there exist upper and lower limits of excitation amplitude for which the second subharmonic exists. The second approximation method yields analytical relations for the existence of the subharmonic. These results are confirmed experimentally in Chapter V.

Electrical networks containing ferroelectric capacitors have properties entirely analogous to networks containing saturable reactors. Subharmonic, irrational, and ferroresonant responses were observed.

It is concluded that the perturbation series and Kryloff second approximation methods are sufficiently accurate for engineering analysis of ferroelectric circuits with the nonlinearity of present ceramic dielectrics. These analytical methods satisfactorily predict and describe the phenomena observed and give reasonable numerical results. The questionable approximation of a hysteresis loop by a polynomial renders a more accurate analytical solution of dubious practical value.

It is the opinion of this author that the successive approximation method is the most useful of the analytical methods considered. In some problems, where a first approximation solution is sufficient and the algebraic complexity great, the equivalent linearization method is preferable. In the cases analyzed in detail the Kryloff second approximation yielded results consistent with results of other analytical methods and seems to be more straightforward to apply than perturbation series.

The similarity of starting conditions and responses for the nonlinear circuits discussed herein and for regenerative dividers suggests that these are much more closely related than is generally recognized. Some properties similar to those of synchronized oscillators were observed. It is suggested that these similarities be considered in further studies in order to establish a fundamental pattern and common properties for these nonlinear circuits.

APPENDIX

A method of obtaining the second approximation for a system of two degrees of freedom follows. Consider a pair of quasilinear second order differential equations in normal form given by

$$x_1'' + \left(\frac{r}{s}\right)^2 x_1 = \epsilon\, h_1(x_1, x_2, x_1', x_2') + \epsilon^2\, h^2(x_1, x_2, x_1', x_2')$$

$$+ \epsilon\, H \sin n\,\theta$$

and

$$x_2'' + \left(\frac{\ell}{k}\right)^2 x_2 = \epsilon\, g_1(x_1, x_2, x_1', x_2') + \epsilon^2\, g_2(x_1, x_2, x_1', x_2')$$

$$+ \epsilon\, G \sin n\,\theta \quad ,$$

where h_1 and g_1 are polynomials in x_1, x_2 and their derivatives. The primes are used to indicate differentiation with respect to $\theta = \omega t$. Terms of third or higher order of ϵ are neglected in this second approximation.

Solutions will be sought of the form

$$x_1 = w(X_1, X_2, \phi_1, \phi_2, \theta) = w_o + \epsilon\, w_1 + \epsilon^2\, w_2$$

$$x_2 = y(X_1, X_2, \phi_1, \phi_2, \theta) = y_o + \epsilon\, y_1 + \epsilon^2\, y_2,$$

where w_o, w_1, - - - y_o - - - y_2 are dependent on the amplitudes and phases X_1, X_2, ϕ_1, ϕ_2 as well as the independent variable θ. Let it be assumed that the one resonant frequency $\Omega_1/2\pi$ is near $(r/s)\omega$ and the other resonance is near $\frac{\ell}{k}\omega$. That is, $\Omega_1 \doteq \frac{r}{s}\,\omega$ and $\Omega_2 \doteq \frac{\ell}{k}\,\omega$, where $\frac{r}{s}$ and $\frac{\ell}{k}$ are rational fractions for subharmonic and harmonic responses, but may be irrational. Then the total instantaneous phase angles of x_1 and x_2 can be expressed as

$$\psi_1 = \frac{r}{s}\,\omega t + \phi_1 = \frac{r}{s}\,\theta + \phi_1$$

and

$$\psi_2 = \frac{\ell}{k}\,\omega t + \phi_2 = \frac{\ell}{k}\,\theta + \phi_2 \quad .$$

Further the amplitudes X_1, X_2 and phases ϕ_1, ϕ_2 will be defined as

$$\omega \frac{dX_1}{d\theta} = \frac{dX_1}{dt} = A\,(X_1, X_2, \phi_1, \phi_2) = \epsilon A_1 + \epsilon^2 A_2$$

$$\omega \frac{d\phi_1}{d\theta} = \frac{d\phi_1}{dt} = \sigma\,(X_1, X_2, \phi_1, \phi_2) = \epsilon \sigma_1 + \epsilon^2 \sigma_2$$

$$\omega \frac{dX_2}{d\theta} = \frac{dX_2}{dt} = B\,(X_1, X_2, \phi_1, \phi_2) = \epsilon B_1 + \epsilon^2 B_2$$

$$\omega \frac{d\phi_2}{d\theta} = \frac{d\phi_2}{dt} = \gamma(X_1, X_2, \phi_1, \phi_2) = \epsilon \gamma_1 + \epsilon^2 \gamma_2$$

Differentiation of $w(a_1 - - - \theta)$ by the chain rule yields

$$\frac{dX_1}{d\theta} = \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial X_1}\frac{\partial X_1}{\partial \theta} + \frac{\partial w}{\partial X_2}\frac{\partial X_2}{\partial \theta} + \frac{\partial w}{\partial \phi_1}\frac{\partial \phi_1}{\partial \theta} + \frac{\partial w}{\partial \phi_2}\frac{\partial \phi_2}{\partial \theta}$$

$$= \frac{\partial w}{\partial \theta} + \frac{A}{\omega}\frac{\partial w}{\partial X_1} + \frac{B}{\omega}\frac{\partial w}{\partial X_2} + \frac{\sigma}{\omega}\frac{\partial w}{\partial \phi_1} + \frac{\gamma}{\omega}\frac{\partial w}{\partial \phi_2}$$

and

$$\frac{d^2x_1}{d\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{dx_1}{d\theta}\right) + \frac{\partial}{\partial X_1}\left(\frac{dx_1}{d\theta}\right)\frac{\partial X_1}{\partial\theta} + \frac{\partial}{\partial X_2}\left(\frac{dx_1}{d\theta}\right)\frac{\partial X_2}{\partial\theta}$$

$$+ \frac{\partial}{\partial\phi_1}\left(\frac{dx_1}{d\theta}\right)\frac{\partial\phi_1}{\partial\theta} + \frac{\partial}{\partial\phi_2}\left(\frac{dx_1}{\partial\theta}\right)\frac{\partial\phi_2}{\partial\theta}$$

$$= \frac{\partial^2 w}{\partial\theta^2} + \frac{A}{\omega}\frac{\partial^2 w}{\partial\theta\partial X_1} + \frac{B}{\omega}\frac{\partial^2 w}{\partial\theta\partial X_2} + \frac{\sigma}{\omega}\frac{\partial^2 w}{\partial\theta\partial\phi_1} + \frac{\gamma}{\omega}\frac{\partial^2 w}{\partial\theta\partial\phi_2}$$

$$+ \frac{A}{\omega}\Bigg(\frac{\partial^2 w}{\partial X_1\partial\theta} + \frac{A}{\omega}\frac{\partial^2 w}{\partial X_1^2} + \frac{1}{\omega}\frac{\partial w}{\partial X_1}\frac{\partial A}{\partial X_1} + \frac{B}{\omega}\frac{\partial^2 w}{\partial X_1\partial X_2} + \frac{1}{\omega}\frac{\partial w}{\partial X_2}\frac{\partial B}{\partial X_1}$$

$$+ \frac{\sigma}{\omega}\frac{\partial^2 w}{\partial X_1\partial\phi_1} + \frac{1}{\omega}\frac{\partial w}{\partial\phi_1}\frac{\partial\sigma}{\partial X_1} + \frac{\gamma}{\omega}\frac{\partial^2 w}{\partial X_1\partial\phi_2} + \frac{1}{\omega}\frac{\partial w}{\partial\phi_2}\frac{\partial\gamma}{\partial X_1}\Bigg)$$

$$+ \frac{B}{\omega}\Bigg(\frac{\partial^2 w}{\partial X_2\partial\theta} + \frac{A}{\omega}\frac{\partial^2 w}{\partial X_1\partial X_2} + \frac{1}{\omega}\frac{\partial w}{\partial X_1}\frac{\partial A_2}{\partial X_2} + \frac{B}{\omega}\frac{\partial^2 w}{\partial X_2^2} + \frac{1}{\omega}\frac{\partial w}{\partial X_2}\frac{\partial B}{\partial X_2}$$

$$+ \frac{\sigma}{\omega}\frac{\partial^2 w}{\partial X_2\partial\phi_1} + \frac{1}{\omega}\frac{\partial w}{\partial\phi_1}\frac{\partial\sigma}{\partial X_2} + \frac{\gamma}{\omega}\frac{\partial^2 w}{\partial X_2\partial\phi_2} + \frac{1}{\omega}\frac{\partial w}{\partial\phi_2}\frac{\partial\gamma}{\partial X_2}\Bigg)$$

$$+ \frac{\sigma}{\omega}\Bigg(\frac{\partial^2 w}{\partial\phi\partial\theta} + \frac{A}{\omega}\frac{\partial^2 w}{\partial\phi_1\partial X_1} + \frac{1}{\omega}\frac{\partial w}{\partial X_1}\frac{\partial A_1}{\partial\phi_1} + \frac{B}{\omega}\frac{\partial^2 w}{\partial\phi_1\partial X_2} + \frac{1}{\omega}\frac{\partial w}{\partial X_2}\frac{\partial B}{\partial\phi_1}$$

$$+\frac{\sigma}{\omega}\frac{\partial^2 w}{\partial \phi_1^{\,2}}+\frac{1}{\omega}\frac{\partial w}{\partial \phi_1}\frac{\partial \sigma}{\partial \phi_1}+\frac{\gamma}{\omega}\frac{\partial^2 w}{\partial \phi_1 \partial \phi_2}+\frac{1}{\omega}\frac{\partial w}{\partial \phi_2}\frac{\partial \gamma}{\partial \phi_1}\Bigg)$$

$$+\frac{\gamma}{\omega}\Bigg(\frac{\partial^2 w}{\partial \phi_2 \partial \theta}+\frac{A}{\omega}\frac{\partial^2 w}{\partial \phi_2 \partial X_1}+\frac{1}{\omega}\frac{\partial w}{\partial X_1}\frac{\partial A_1}{\partial \phi_2}+\frac{B}{\omega}\frac{\partial^2 w}{\partial \phi_2 \partial X_2}$$

$$+\frac{1}{\omega}\frac{\partial w}{\partial X_2}\frac{\partial B}{\partial \phi_2}+\frac{\sigma}{\omega}\frac{\partial^2 w}{\partial \phi_2 \partial \phi_1}+\frac{1}{\omega}\frac{\partial w}{\partial \phi_1}\frac{\partial \sigma}{\partial \phi_1}+\frac{\gamma}{\omega}\frac{\partial^2 w}{\partial \phi_2^{\,2}}$$

$$+\frac{1}{\omega}\frac{\partial w}{\partial \phi_2}\frac{\partial \gamma}{\partial \phi_2}\Bigg).$$

Now the expansions of ω, A, B, σ and γ in powers of ϵ are to be substituted into these equations, and the resulting x_1, x_1, x_1 equations substituted into the original differential equation for x_1. If these steps are carried out the second order differential equation for x_1 contains powers of ϵ on each side of the equality. If the coefficients of equal powers of ϵ of the right and left hand sides are equated, there results a system of second order linear differential equations for the zeroth, first and second approximations w_o, w_1 and w_2. Since the zeroth approximation for x_1 is

$$w_o = X_1 \sin\left(\frac{r}{s}\,\theta + \phi_1\right) ,$$

then

$$\frac{\partial w_o}{\partial X_2} = \frac{\partial w_o}{\partial \phi_2} = \frac{\partial^2 w_o}{\partial X_1^{\,2}} = 0 .$$

The differential equations, which define the solution to the zeroth, first and second approximations, are

$$\frac{\partial^2 w_o}{\partial \theta^2} + \left(\frac{r}{s}\right)^2 w_o = 0$$

$$\frac{\partial^2 w_1}{\partial \theta^2} + \left(\frac{r}{s}\right)^2 w_1 = H \sin n\,\theta + h_1\left(w_o, y_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}\right) - 2\frac{A_1}{\omega}\frac{\partial^2 w_o}{\partial \theta \partial X_1} - 2\frac{\sigma_1}{\omega}\frac{\partial^2 w_o}{\partial \theta \partial \phi_1}$$

$$\begin{aligned}
\frac{\partial^2 w_2}{\partial \theta^2} + \left(\frac{r}{s}\right)^2 w_2 = & - 2\frac{A_2}{\omega}\frac{\partial^2 w_o}{\partial \theta \partial X_1} - 2\frac{A_1}{\omega}\frac{\partial^2 w_1}{\partial \theta \partial X_1} - 2\frac{B_1}{\omega}\frac{\partial^2 w_1}{\partial \theta \partial X_2} \\
& - 2\frac{\sigma_2}{\omega}\frac{\partial^2 w_o}{\partial \theta \partial \phi_1} - 2\frac{\sigma_1}{\omega}\frac{\partial^2 w_1}{\partial \theta \partial \phi_1} - 2\frac{\gamma_1}{\omega}\frac{\partial^2 w_1}{\partial \theta \partial \phi_2} \\
& - \frac{A_1}{\omega^2}\left(\frac{\partial w_o}{\partial X_1}\frac{\partial A_1}{\partial X_1} + \sigma_1\frac{\partial^2 w_o}{\partial X_1 \partial \phi_1} + \frac{\partial w_o}{\partial \phi_1}\frac{\partial \sigma_1}{\partial X_1}\right) \\
& - \frac{B_1}{\omega^2}\left(\frac{\partial w_o}{\partial X_1}\frac{\partial A_1}{\partial X_2} + \frac{\partial w_o}{\partial \phi_1}\frac{\partial \sigma_1}{\partial X_2}\right) - \frac{\gamma_1}{\omega^2}\left(\frac{\partial w_o}{\partial X_1}\frac{\partial A_1}{\partial \phi_2} + \frac{\partial w_o}{\partial \phi_1}\frac{\partial \sigma_1}{\partial \phi_2}\right) \\
& - \frac{\sigma_1}{\omega^2}\left(A_1\frac{\partial^2 w_o}{\partial \phi_1 \partial X_1} + \frac{\partial w_o}{\partial X_1}\frac{\partial A_1}{\partial \phi_1} + \sigma_1\frac{\partial^2 w_o}{\partial \phi_1^2} + \frac{\partial w_o}{\partial \phi_1}\frac{\partial \sigma_1}{\partial \phi_1}\right) \\
& + h_2\left(w_o, y_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}\right) + \left.\frac{\partial h_1}{\partial X_1}\right|_0 w_1 + \left.\frac{\partial h_1}{\partial X_2}\right|_0 y_1
\end{aligned}$$

$$\left.\frac{\partial h_1}{\partial X_1'}\right|_0 \left(\frac{\partial w_1}{\partial \theta} + \frac{A_1}{\omega}\frac{\partial w_o}{\partial X_1} + \frac{\sigma_1}{\omega}\frac{\partial w_o}{\partial \phi_1}\right) + \left.\frac{\partial h_1}{\partial X_2'}\right|_0 \left(+\frac{\partial y_1}{\partial \theta} + \frac{B_1}{\omega}\frac{\partial y_o}{\partial X_2} + \frac{\gamma_1}{\omega}\frac{\partial y_o}{\partial \phi_2}\right).$$

In the last equation

$$\left.\frac{\partial h_1}{\partial X_1'}\right|_0$$

denotes the derivative of the function $h_1(x_1, x_2, x_1', x_2')$ with respect to x_1', evaluated at the point $w_o, y_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}$.

The equations for y_o, y_1, y_2, the successive approximations to x_2, are similarly found to be

$$\frac{\partial^2 y_o}{\partial \theta^2} + \left(\frac{\ell}{k}\right)^2 y_o = 0$$

$$\frac{\partial^2 y_1}{\partial \theta^2} + \left(\frac{\ell}{k}\right)^2 y_1 = G \sin n\theta + g_1\left(w_o, y_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}\right) - 2\frac{B_1}{\omega}\frac{\partial^2 y_o}{\partial \theta \partial X_2} - 2\frac{\gamma_1}{\omega}\frac{\partial^2 y_o}{\partial \theta \partial \phi_2}$$

$$\frac{\partial^2 y_2}{\partial \theta^2} + \left(\frac{\ell}{k}\right)^2 y_2 = -2\frac{A_1}{\omega}\frac{\partial^2 y_1}{\partial \theta \partial X_1} - 2\frac{B_2}{\omega}\frac{\partial^2 y_o}{\partial \theta \partial X_2} - 2\frac{B_1}{\omega}\frac{\partial^2 y_1}{\partial \theta \partial X_2} - 2\frac{\sigma_1}{\omega}\frac{\partial^2 y_1}{\partial \theta \partial X_2} - 2\frac{\gamma_2}{\omega}\frac{\partial^2 y_o}{\partial \theta \partial \phi_1} - 2\frac{\gamma_1}{\omega}\frac{\partial^2 y_1}{\partial \theta \partial \phi_2} - \frac{A_1}{\omega^2}\left(\frac{\partial y_o}{\partial X_2}\frac{\partial B_1}{\partial X_1} + \frac{\partial y_o}{\partial \phi_2}\frac{\partial \gamma_1}{\partial X_1}\right)$$

$$- \frac{B_1}{\omega^2}\left(\frac{\partial y_o}{\partial X_2}\frac{\partial B_1}{\partial X_2} + \gamma_1 \frac{\partial^2 y_o}{\partial X_2 \partial \phi_2} + \frac{\partial y_o}{\partial \phi_2}\frac{\partial \gamma_1}{\partial X_2}\right)$$

$$- \frac{\sigma_1}{\omega^2}\left(\frac{\partial y_o}{\partial X_2}\frac{\partial B_1}{\partial \phi_1} + \frac{\partial y_o}{\partial \phi_2}\frac{\partial \gamma_1}{\partial \phi_1}\right)$$

$$- \frac{\gamma_1}{\omega^2}\left(B_1 \frac{\partial^2 y_o}{\partial \phi_2 \partial X_2} + \frac{\partial y_o}{\partial X_2}\frac{\partial B_1}{\partial X_2} + \gamma_1 \frac{\partial^2 y_o}{\partial \phi_2^2} + \frac{\partial y_o}{\partial \phi_2}\frac{\partial \gamma_1}{\partial \phi_2}\right)$$

$$+ g_2\left(w_o, y_o, \frac{\partial w_o}{\partial \theta}, \frac{\partial y_o}{\partial \theta}\right) + \left.\frac{\partial g_1}{\partial X_1}\right|_0 w_1 + \left.\frac{\partial g_1}{\partial X_2}\right|_0 y_1$$

$$+ \left.\frac{\partial g_1}{\partial X_1'}\right|_0 \left[\frac{\partial w_1}{\partial \theta} + \frac{A_1}{\omega}\frac{\partial w_o}{\partial X_1} + \frac{\sigma_1}{\omega}\frac{\partial w_o}{\partial \phi_1}\right]$$

$$+ \left.\frac{\partial g_1}{\partial X_2'}\right|_0 \left[\frac{\partial y_1}{\partial \theta} + \frac{B_1}{\omega}\frac{\partial y_o}{\partial X_2} + \frac{\sigma_1}{\omega}\frac{\partial y_o}{\partial \phi_2}\right] .$$

The quantities A_1, A_2, B_1, B_2, σ_1, σ_2, γ_1 and γ_2 are determined from the above differential equations. That is, these quantities are so selected that terms of the right-hand side at the angular frequencies $\frac{r}{s}\,\omega$ and $\frac{\ell}{k}\,\omega$ vanish in the w and y equations respectively. Thus these parameters are defined by the conditions that y_o, y_1, y_2, w_o, w_1 and w_2 be periodic and bounded in time. Once the parameters A_1 - - - $_2$ are known they can be substituted into their corresponding first order differential equations. The equilibrium conditions of this set of four first order equations determine the values of X_1, X_2, ϕ_1 and ϕ_2 which satisfy the original pair of differential equations to the second order of ϵ.

This general second approximation analysis will not be carried further at this time due to its algebraic complexity. A second approximation solution to a system of two degrees of freedom is developed in the section of Chapter V on irrational solutions.

BIBLIOGRAPHY

Literature Cited

1. Heegner, K., "Uber Selbsterregungserscheinungen bei Systemen mit gestorter Superposition," Zeitschrift fur Physik, 29, (1924), 19; 33, (1925), 85.

2. Fallou, J., "Sur un Demultiplication de Frequence Statique," Revue Generale de l'Electricite, 19, (1926) 987.

3. Dehors, M. R., "Recherches sur la Demultiplication de Frequence Ferromagnetique," Revue Generale de l'Electricite (Paris), 56, (1947), 455-67.

4. McCrumm, J. D., "An Experimental Investigation of Subharmonics," Transactions A.I.E.E., 60, (1941) 533-540 and 722.

5. Fallou, J., "Electrical Frequency Changer," United States Patent 1,633,481, (June 21, 1927).

6. Heegner, K., "Electrical Frequency Changer," United States Patent 1,656,195, (January 17, 1928).

7. McCreary, H. J., "Electrical Frequency Changer," United States Patent 2,455,078, (November 30, 1948).

8. Manley, J. M., "Electrical Frequency Changer," United States Patent 2,272,246, (February 10, 1942).

9. "Sub-Cycle," The New Ringer for Exchanges," Telephony, 111, (July 18, 1936), 30.

10. Manley, J. M. and Peterson, E., "Negative Resistance Effects in Saturable Reactor Circuits," Transactions A.I.E.E., 65, (1946) 870-881.

11. Bennett, W. R., "A General Review of Linear Varying Parameter and Non-Linear Circuit Analysis," Proceedings I.R.E., 38, (1950) 259-63.

12. Cartwright, M. L., "Non-Linear Vibrations," The Advancement of Science, 6, No. 21 (April 1949).

13. Minorsky, N., Introduction to Non-Linear Mechanics, J. W. Edwards, Ann Arbor, Michigan, (1947).

14. Wainer, E., "High Titania Dielectrics," Trans. American Electrochemical Society, 89, (1946) 331.

15. Wul, B., "Barium Titanate, A New Ferro-Electric," Nature, (London), 157, (June 15, 1946) 808.

16. von Hippel, A., "Ferroelectricity, Domain Structure, and Phase Transitions of Barium Titanate," Reviews of Modern Physics, 22, (1950), 221.

17. Roberts, S., "Dielectric Constants and Polarizabilities of Loss in Simple Crystals and Barium Titanate," Physical Review, 76, (1949), 1215.

18. Jaynes, E. T., Ferroelectricity, Investigations in Physics Series, Princeton University Press, Princeton, N. J. (1953).

19. Merz, W. J., "The Electric and Optical Behavior of Ba TiO_3 Single Domain Crystals," Physical Review, 76, (1949) 221.

20. Roberts, Shepard, "Barium Titanate and Barium-Strontium Titanate as Nonlinear Dielectrics," Technical Report 1, Laboratory for Insulation Research M.I.T. (December 1946).

21. Vincent, A. M., "Dielectric Amplifier Fundamentals," Electronics, 24, (December 1951) 84-88.

22. Shaw, George S. and Jenkins, J. L., "Characteristics and Applications of Voltage Sensitive Dielectrice," IRE Electronic Components Symposium, Pasadena, California, April 29, 1953.

23. Urkowitz, H., "A Ferroelectric Amplifier," Journal of Franklin Institute, 254, No. 6, (December 1952) 517.

24. Hollman, H. E., "Nonlinear Elements and Applications in AF and RF Circuits," Tele-Tech, 11, (April and May 1952), 46, 56.

25. Tucker, D. G., "Non-Linear Inductance and Capacitance as Modulators for Amplitude Modulation Systems," Electrical Engineer's Journal, 42, (October 1949), 156-159.

26. Dranetz, A. I., Howatt, G. N. and Crownover, J. W., "Barium Titanates as Circuit Elements," Tele-Tech, 8, (April, May and June 1949), 29-31, 53-59, 28-30, 54, 56, 57, 36, 39, 52 and 53.

27. Anderson, J. W., "Ferroelectric Materials as Computer Storage Element," Transactions A.I.E.E., Pt. 1, Communications Electronics, 72, (January 1953) No. 4, 395.

28. Littman, M. F., "Ultrathin Tapes of Magnetic Alloys with Rectangular Hysteresis Loops," Transactions A.I.E.E., 71, Pt. 2 (1952), 220.

29. Goldsmith, H. A., Snyder, C. L., Albers-Schoenberg, E., "Magnetic Ferrities--Core Materials for High Frequencies," Electrical Manufacturing, 44, (December, 1949), 86.

30. Brockman, F. G., "Magnetic Ferrities for High-Frequency Uses," Electronic Engineering (London), 68, (December 1949), 1077-1080.

31. Cartwright, M. L., "Topological Aspect of Forced Oscillations," Research, 1, (1948), 601.

32. Cartwright, M. L., "Forced Oscillations in Nearly Sinusoidal Systems," Journal I.E.E. (London), 95, Pt. III, 88, (1948).

33. Schouten, J. P. and Heyin, "Subharmonic Oscillations in Electric Circuits Containing Iron-Core Reactors," Applied Scientific Research, Section B, 2, No. 4 (1942), 301.

34. Klotter, K., "Steady State Vibrations in Systems having Arbitrary Damping Forces," paper presented at 1953 Symposium on Nonlinear Circuit Analysis, New York, New York.

35. Stoker, J. J., Nonlinear Vibrations, New York, New York, Interscience Publishers, (1950).

36. McLachlan, N. W., Ordinary Non-Linear Differential Equations, London, England, Oxford University Press, (1950).

37. Kryloff, N.,. Bogoliuboff, N., Introduction to Nonlinear Mechanics, Translated by S. Lefschetz, Princeton University Press, Princeton, New Jersey (1943).

38. Reuter, G. E. H., "Subharmonics in Non-Linear System with Unsymmetrical Restoring Force," Quarterly Journal Mechanics and Applied Mathematics, 2, (1949), 198.

39. Friedrichs, K. O., Stoker, J. J., "Forced Vibrations with Non-Linear Restoring Force," Quarterly of Applied Mathematics, 1, (1943), 97.

40. Levensen, M. E., "Harmonic and Subharmonic Response for the Duffing Equation," Journal Applied Physics, 20, (November 1949), 1045-1051.

41. Minorsky, N., op. cit., p. 308.

42. Bruce, Victor G., "Investigations of a Mechanical System with Periodic Parameters and a Nonlinear Restoring Force," Stanford University doctoral disseration, (1950).

43. Bothwell, F. E., "Transients in Multiple Periodic Nonlinear Systems," Quarterly Applied Mathematics, 8 (1950), 247-54.

44. Ludeke, C. A., "A Method of Equivalent Linearization for Non linear Oscillatory Systems with Large Nonlinearity," Journal Applied Physics, 20, (July 1949), 694-699.

45. Ludeke, C. A., "Predominately Subharmonic Oscillations," Journal Applied Physics, 22, (November 1951), 1321-26.

46. Malkin, I. G., "Oscillations of Systems with One Degree of Freedom, Close to Systems of Lyapunov," American Mathematical Society Translation, No. 22, (1950).

47. Obi, Chic, "Periodic Solutions of Nonlinear Differential Equations of the Second Order--IV," Proc. Cambridge Philosophical Society, 47, Pt. 4, (October 1951), 741-751.

48. Stoker, op. cit., p. 98.

49. Minorsky, op. cit., p. 293.

50. Minorsky, op. cit., p. 229.

51. Bulgakov, B. V., "On the Method of Van der Pol and its Application to Nonlinear Systems," Journal Franklin Institute, 241, (1946), 31-54.

52. Liapounoff, M. A., "Probléme Général de la Stabilité du Mouvement," Annals of Mathematics, Series No. 17, Princeton, New Jersey, Princeton University Press, (1948).

53. Bothwell, F. E., "The Current Status of Dynamic Stability Theory," Transactions A.I.E.E., 71, Pt. 1 Communications and Electronics, (July 1952), 223.

54. Malkin, I. G., "Certain Questions on the Theory of the Stability of Motion in the Sense of Liapounoff," American Mathematical Trans., No. 20 (1950).

55. Schaffner, Johannes A., "Almost Sinusoidal Oscillations in Nonlinear Systems--1. Introduction--Simultaneous Oscillation," University of Illinois--Engineering Experiment Station Bulletin Series, No. 395, (1951).

56. de Bretteville, A., "Oscillographic Study of the Dielectric Properties of Barium Titanate," Journal American Ceramic Society, 29, (1946) 303-7.

57. Roberts, S., op.cit., p. 97.

58. Urkowitz, H., op.cit.

59. McKune, W. J. and Burst, M. F., "Subharmonics in a Series Nonlinear Circuit," Transactions A.I.E.E., Pt. 1 Communication and Electronics, 71, (July 1952) 200.

60. Stoker, J. J., op.cit., p. 109.

61. Stoker, J. J., "Mathematical Characteristics of Nonlinear Phenomena," paper presented at 1953 Symposium on Nonlinear Circuit Analysis, New York, New York.

62. Cohen, Hirsh G., Subharmonic Synchronization for the Forced van der Pol Equation, Doctoral dissertation, Brown University (1951).

Other References

Anderson, P. W., "Theory of Ferroelectric Behavior of Barium Titanate," Ceramic Age, 57, (April 1951), 29.

Angello, Stephan J., "The Effect of Initial Conditions on Subharmonic Currents in Non-Linear Circuits," Transactions A.I.E.E., 61, (1942) 625-627.

Cunningham, W. J., "The Growth of Subharmonic Oscillations," Journal American Acoustical Society, 23, (1951), 418-422.

Drake, B., "Ferroresonance," Electronic Engineering, 21, No. 254 (April 1949), 135-9.

Friedlander, F. G., "On the Forced Vibrations of Quasi-linear Systems," Quarterly Journal Mathematics and Applied Mechanics, Vol. III, (1950) 364-376.

Friedlander, F. G., "On the Recurrent Solutions of a Class of Non-linear Differential Equations," Proceedings Cambridge Philosophical Society, 47, (1951), 315-330.

Gadsen, C. P., "An Electrical Network with Varying Parameters," Quarterly of Applied Mathematics, 8, (1950), 199.

Graffi, D., "Forced Oscillations for Several Nonlinear Circuits," Annals of Mathematics, 54, (September 1951), 262-271.

Haag, Jules, "Sur la Synchronization des Systems a Plusieurs Degrés de Liberte," E'cole Normale Superieure Annales Scientifiques (Paris), 64 (1947).

Haag, Jules, "Sur L' Existence et la Stabilite des Solutions Periodiques de certains Systemes Differentials," 65, (1948), 298-335.

Hartley, R. V. L., "Oscillations in Systems with Non-Linear Reactance," Bell System Technical Journal, 15, (1936), 424-440.

Hoffmann, G., "Harmonic and Combination Oscillations in Ferromagnetic Materials," Archiv der Elektristhen Ubertragung, 6, (March 1952), 99-108.

Hsia, P. S., "Graphical Analysis for Non-Linear Systems," Proceedings of I.E.E. (London), 99, Pt. 2, No. 68, (April 1952), 125-34.

Karman, T. Don, "The Engineer Grapples with Non-Linear Problems," Bulletin American Mathematical Society, 46, (1940), 615-684.

Keller, E. G., "Analytical Methods of Solving Discrete Nonlinear Problems in Electrical Engineering," Transactions A.I.E.E., 60, (1941), 1194.

Klotter, K., and Pinney, E., "Stability Criteria for Forced Vibrations in Nonlinear Systems," Journal of Applied Mechanics, 20, No. 1, (March 1953), 9.

Kornetzki, M., "Results of Measurements on High Permeability Ferrite Cores," Zeitschrift fur Angewandte Physik, 3, (January 1951), 5-9.

Kornetzki, M., "Eine Bezichung zwischen Dielektrizitatskonstante und Verlustivinkel ferroelektrischer Staffe," Zeitschrift fur Angewandte Physik, 2, (1950), 446-8.

Lee Wu, M. A., "Subharmonic Resonance of System Having Non-Linear Spring with Variable Coefficient," American Society Mechanical Engineering, Proceedings of First U. S. National Congress of Applied Mechanics, (1952).

Lefschetz, S., "Contributions to the Theory of Non-Linear Oscillations," Annals of Mathematics, No. 20, 29, Princeton University Press.

McCreary, H. J., "The Magnetic-Cross Value and its Application to Sub-frequency Power Generation," Proceedings National Electronics Conference, 5, (1949), 450-466.

Obi, Chike, "Subharmonic Solutions of Non-Linear Differential Equations of the Second Order," Journal London Mathematical Society, 25, (1950), 217-226.

Roberts, S. and Thurnauer, H., "Digest of Literature on Dielectrics," National Research Council, Vol. XIV, (1951).

Roberts, S., "Dielectric and Piezoelectric Properties of Barium Titanate," Physics Review, 71, No. 13, (June 15, 1947), 890-895.

Rosenhamer, H., "Contribution to the Mathematical Treatment of Non-Linear Phenomena," Schweizerischer Elektrotechnischer Verin, Bulletin, 40, (January 8, 1949), 5-18.

Rouelle, E., "Quelques Nouvelles Expexences Demultiplication de Frequences dans un circuit oscillant dont La Bobine est a Nayau de Fer," Revue Generale del Electricite (Paris, France), 40, (December 1936), 811.

Rudenberg, R., "Nonharmonic Oscillations as Caused by Magnetic Saturation," Transactions A.I.E.E., 68, Pt. 1, (1949), 676-685.

"References on Piezoelectric Titanates," "High Dielectric Ceramics," Booklets published by Titanium Alloy Manufacturing Division National Lead Company, Niagara Falls, New York.

Travis, Irven and Weygandt, C. N., "Subharmonic in Circuits Containing Iron Cored Reactors," Transactions A.I.E.E., 57, (1938), 423-431.

Travis, I., "Subharmonic Circuits Containing Iron Cored Inductors--II," Transactions A.I.E.E., 58, (1939), 735-742.

Vander Ziel, A., "On the Mixing Properties of Non-Linear Condensers," Journal of Applied Physics, 19, (1948), 999.

Wenke, W., "Die Instabilitat linearer und nicht linearer Schwinger (Mitnahme schevingungen)," Hochfrequentztechnik und Electroakustik, 55, (1940), 94-101 and 109-120.

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